Practice Midterm

Problem 1: a) Let V be a subset of $\mathbb{R}^n$. When is V called a *subspace* of $\mathbb{R}^n$? Give a precise definition! (3 pts)

b) Consider the subset V of $\mathbb{R}^4$ consisting of all vectors (x_1, x_2, x_3, x_4) in $\mathbb{R}^4$ such that $x_1 \geq -1$, that is,

$$V = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 \geq -1\}.$$

Is V a subspace of $\mathbb{R}^4$? Justify your answer! (2 pts)

c) Suppose U and V are subsets of $\mathbb{R}^n$. Then the intersection $U \cap V$ of U and V is defined to be the set of all vectors x in $\mathbb{R}^n$ that belong both to U and V, that is,

$$U \cap V = \{x \in \mathbb{R}^n : x \in U \text{ and } x \in V\}.$$

Show that if U and V are subspaces of $\mathbb{R}^n$, then $U \cap V$ is also a subspace of $\mathbb{R}^n$. (5 pts)

Problem 2: Use the method of least squares to find the optimal fit of the data points $(0,2)$, $(1,1)$, $(2,4)$, $(3,3)$ by a line in the x-y-plane. (10 pts)

Problem 3: An $(n \times n)$-matrix A is called *nilpotent* if there exists $k \in \mathbb{N}$ such that $A^k = \mathbf{0}$. Here $\mathbf{0}$ denotes the $(n \times n)$-matrix whose all entries are equal to 0.

a) Give an example of a (3×3)-matrix $A \neq \mathbf{0}$ that is nilpotent. (5 pts)

b) Show that no nilpotent matrix A is invertible. (5 pts)

Hint: Argue by "contradiction"; that is, assume that there exists an invertible nilpotent matrix A. Derive a consequence that you know is false and conclude that your hypothesis (namely, that there exists an invertible nilpotent matrix) must also be false.

Problem 4: Let $v_1, \ldots, v_k$ be orthonormal vectors in $\mathbb{R}^n$. Show that then the vectors $v_1, \ldots, v_k$ are linearly independent. (5 pts)

Problem 5: Let V be a subspace of $\mathbb{R}^n$. Recall that the *orthogonal complement* $V^\perp$ of V is defined to be the set of all vectors $u \in \mathbb{R}^n$ that are orthogonal to *all* vectors in V, i.e.,

$$V^\perp = \{u \in \mathbb{R}^n : u \cdot v = 0 \text{ for all } v \in V\}.$$

We choose a basis $v_1, \ldots, v_k$ of V, and a basis $u_1, \ldots, u_l$ of $V^\perp$.

a) Show that the vectors $v_1, \ldots, v_k, u_1, \ldots, u_l$ are linearly independent. (3 pts)

b) Show that every vector $x \in \mathbb{R}^n$ can be uniquely represented as $x = p + p^\perp$, where $p \in V$ and $p^\perp \in V^\perp$. (3 pts)

c) Show that the vectors $v_1, \dots, v_k, u_1, \dots, u_l$ form a basis of $\mathbb{R}^n$. (2 pts)

d) Show that $\dim V + \dim V^\perp = n$. (2 pts)

Problem 6: Let A be an $(n \times n)$-matrix and suppose that $Ax = x$ for all $x \in \mathbb{R}^n$. Show that then A is equal to the unit matrix I_n. (10 pts)