Homework 4 (due: Fr, 5/1)

Problem 1: Let $(S, \mathcal{S})$ be a Lusin space, $p\colon S \times \mathcal{S} \to [0,1]$ be a Markov kernel, and μ be a probability measure on S. Suppose that X_n, $n \in \mathbb{N}_0$, is the canonical Markov chain for p and the initial distribution μ defined on $(S^\infty, \mathcal{S}_\infty, \mathbb{P}_\mu)$ as the underlying probability space (so that $X_n\colon S^\infty \to S$ is the projection map onto the n-th coordinate).

Suppose Y_n, $n \in \mathbb{N}_0$, is another Markov chain for p and μ defined on some other probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathcal{F}_n$, $n \in \mathbb{N}_0$.

a) Show that the map $T\colon \Omega \to S^\infty$ defined as

$$T(\omega) = (Y_0(\omega), Y_1(\omega), \dots) \in S^\infty \quad \text{for } \omega \in \Omega$$

is measurable, i.e., $T^{-1}(M) \in \mathcal{F}$ for each $M \in \mathcal{S}_\infty$.

b) Show that $T_*(\mathbb{P}) = \mathbb{P}_\mu$.

c) Show that if $n \in \mathbb{N}_0$ and $A \subseteq S^n$ is measurable, then

$$\mathbb{P}((Y_0, \dots, Y_{n-1}) \in A) = \mathbb{P}_\mu((X_0, \dots, X_{n-1}) \in A).$$

Problem 2: Let X_n, $n \in \mathbb{N}_0$, be a Markov chain with state space $\mathbb{R}^n$ defined on some underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathcal{F}_n$, $n \in \mathbb{N}_0$. Suppose that the Markov kernel $p\colon \mathbb{R}^n \times \mathcal{B}_n \to [0,1]$ is translation invariant in the sense that

$$p(x_0 + x, x_0 + A) = p(x, A)$$

whenever $x, x_0 \in \mathbb{R}^n$ and $A \in \mathcal{B}_n$.

Show that then for each $n \in \mathbb{N}$ the random variables

$$X_0,\ \xi_1 = X_1 - X_0, \dots, \xi_n = X_n - X_{n-1}$$

are independent and $\xi_1, \dots, \xi_n$ are identically distributed.