Homework 3 (due: Fr, 3/24)

Problem 1: Let \((\Omega, \mathcal{F}, P)\) be a probability space, \((S, \mathcal{S})\) be a measurable space, \(\mathcal{F}_n, n \in \mathbb{N}_0\), be a filtration on \((\Omega, \mathcal{F}, P)\), and \(X_n, n \in \mathbb{N}_0\), be an adapted \(S\)-valued process.

a) Suppose that \(A \in \mathcal{S}\) and \(T\) is a stopping time. Show that
\[ \tilde{T} = \inf \{ n > T : X_n \in A \} \]
is also a stopping time.

b) Suppose that \(A \in \mathcal{S}\), and define \(T^k\) inductively as \(T^0 = 0\) and
\[ T^k = \inf \{ n > T^{k-1} : X_n \in A \} \quad \text{for} \ k \geq 1. \]
Show that \(T^k\) for \(k \in \mathbb{N}_0\) is a stopping time.

Problem 2: Let \(G\) be a countable group considered as a state space of a Markov chain. We denote by \(e\) the unit element in \(G\) and by \(xy\) the composition of \(x, y \in G\).

If \(g \in G\) we denote by \(L_g\) the left-translation on \(G\) defined as \(L_g(x) = gx\) for \(x \in G\). This map induces a left-translation map on the infinite product \(G^\infty\) by applying it coordinate-wise. We denote this map also by \(L_g\).

As usual, we denote by \(P_e\) and \(P_g\) the probability measures corresponding to the Markov chains starting at \(e\) and \(g\), respectively. Show that if the underlying Markov kernel \(p: G \times \mathcal{P}(G) \to [0, 1]\) satisfies \(p(gx, gA) = p(x, A)\) for all \(g, x \in G\) and \(A \subseteq G\), where \(gA = \{ ga : a \in A \}\), then \((L_g)_*(P_e) = P_g\).

Problem 3: Consider simple random walk on a connected graph \(G = (V, E)\).

a) Show that if \(v \in V\) is recurrent, then each \(u \in V\) is recurrent.

b) Show that if \(V\) is finite, then every \(v \in V\) is recurrent.