Homework 3 (due: Fr, 3/24)

Problem 1: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(S, \mathcal{S})$ be a measurable space, $\mathcal{F}_n$, $n \in \mathbb{N}_0$, be a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$, and X_n, $n \in \mathbb{N}_0$, be an adapted S-valued process.

a) Suppose that $A \in \mathcal{S}$ and T is a stopping time. Show that
$$\widetilde{T} = \inf\{n > T : X_n \in A\}$$
is also a stopping time.

b) Suppose that $A \in \mathcal{S}$, and define T^k inductively as $T^0 = 0$ and
$$T^k = \inf\{n > T^{k-1} : X_n \in A\} \quad \text{for } k \geq 1.$$
Show that T^k for $k \in \mathbb{N}_0$ is a stopping time.

Problem 2: Let G be a countable group considered as a state space of a Markov chain. We denote by e the unit element in G and by xy the composition of $x, y \in G$.

If $g \in G$ we denote by L_g the left-translation on G defined as $L_g(x) = gx$ for $x \in G$. This map induces a left-translation map on the infinite product G^∞ by applying it coordinate-wise. We denote this map also by L_g.

As usual, we denote by $\mathbb{P}_e$ and $\mathbb{P}_g$ the probability measures corresponding to the Markov chains starting at e and g, respectively. Show that if the underlying Markov kernel $p\colon G \times \mathcal{P}(G) \to [0,1]$ satisfies $p(gx, gA) = p(x, A)$ for all $g, x \in G$ and $A \subseteq G$, where $gA = \{ga : a \in A\}$, then $(L_g)_*(\mathbb{P}_e) = \mathbb{P}_g$.

Problem 3: Consider simple random walk on a connected graph $G = (V, E)$.

a) Show that if $v \in V$ is recurrent, then each $u \in V$ is recurrent.

b) Show that if V is finite, then every $v \in V$ is recurrent.