Homework 1 (due: Fr, 3/10)

Problem 1: Suppose $(X, \mathcal{A})$, $(Y, \mathcal{B})$, $(Z, \mathcal{C})$ are measurable spaces. In the following products of these spaces are equipped with the corresponding product σ-algebras.

Let $f\colon X \times Y \times Z \to \mathbb{R}$ be a bounded measurable function and $p\colon Y \times \mathcal{C} \to [0,1]$ be a Markov kernel (i.e., $C \in \mathcal{C} \mapsto p_y(C) := p(y, C)$ defines a probability measure for each $y \in Y$ and $y \in Y \mapsto p(y, C)$ is a measurable function for each $C \in \mathcal{C}$).

Show that then the function $g\colon X \times Y \to \mathbb{R}$ defined as

$$g(x, y) = \int f(x, y, z)\, dp_y(z) \text{ for } (x, y) \in X \times Y$$

is also measurable.

Hint: Apply the π-λ-theorem and the monotone class theorem.

Problem 2: Let $(S, \mathcal{S})$ be a Lusin space, $p\colon S \times \mathcal{S} \to [0,1]$ be a Markov kernel, and for $x \in S$ let $\mathbb{P}_x$ be the unique probability measure on S^∞ representing the Markov chain with the kernel p and the initial distribution $\mu = \delta_x$.

For $n \in \mathbb{N}$ let $\pi_n\colon S^\infty \to S^n$ be the projection onto the first n coordinates and $\mu_n^x = (\pi_n)_*(\mathbb{P}_x)$ be the n-th marginal of $\mathbb{P}_x$. We denote by $\mathcal{S}_\infty$ and $\mathcal{S}_n$ the Borel σ-algebras on S^∞ and S^n, respectively.

a) Show that if $n \in \mathbb{N}$ and $A \in \mathcal{S}_n$, then the function $x \in S \mapsto \mu_n^x(A)$ is measurable.

b) Show that if $A \in \mathcal{S}_\infty$, then the function $x \in S \mapsto \mathbb{P}_x(A)$ is measurable.

c) Show that if Z is bounded random variable defined on S^∞, then the function $x \in S \mapsto \mathbb{E}_x(Z)$ is measurable.