Midterm (due: Mo, 5/18)

Problem 1: Let X be a $\mathbb{Z}^d$-valued random variable, where $d \in \mathbb{N}$, and let $\varphi(u) = \mathbb{E}(e^{i(u \cdot X)})$, $u \in \mathbb{R}^d$, be its characteristic function. Suppose that X_n, $n \in \mathbb{N}$, are i.i.d. random variables with the same distribution as X. We consider the random walk on $\mathbb{Z}^d$ defined as $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$ for $n \in \mathbb{N}$.

(a) Show that for $r \in [0, 1)$ we have

$$\sum_{n=0}^{\infty} \mathbb{P}(S_n = 0) r^n = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \mathrm{Re}\left(\frac{1}{1 - r\varphi(u)}\right) du.$$

(2pts)

(b) Show that S_n is recurrent (in the sense that 0 is a recurrent state for the Markov chain S_n) if and only if

$$\lim_{r \to 1^-} \int_{[-\pi,\pi]^d} \mathrm{Re}\left(\frac{1}{1 - r\varphi(u)}\right) du = +\infty.$$

(2pts)

(c) Suppose that $d = 1$ and $\varphi'(0) = 0$. Show that S_n is recurrent. (4pts)

(d) Suppose that $d = 1$ and $\mathbb{E}(|X|) < \infty$. Show that S_n is recurrent if and only if and $\mathbb{E}(X) = 0$. (2pts)

(e) Suppose that $d = 2$, $\mathbb{E}(|X|^2) < \infty$, and $\mathbb{E}(X) = 0$. Show that S_n is recurrent. (2pts)

Problem 2: We use the same notation as in Problem 1. We assume that $d = 1$, $\mathbb{E}(|X|) < \infty$, and $\mathbb{E}(X) = 0$. We also assume that there is no proper subgroup of $\mathbb{Z}$ that contains the support of the distribution $\mu = X_*(\mathbb{P})$ of X.

(a) Denote by $N(x) = \#\{n \in \mathbb{N}_0 : S_n = x\}$ the number of visits of the random walk to $x \in \mathbb{Z}$. Show that we have $N(x) = +\infty$ for all $x \in \mathbb{Z}$ almost surely. (2pts)

(b) Let $T := \inf\{n > 0 : S_n = 0\}$ be the first time that the random walk returns to 0. Then T is a stopping time with respect to the filtration $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$, $n \in \mathbb{N}_0$. Show that $T < +\infty$ almost surely. (2pts)

(c) Let $N_T(x) = \#\{0 \le n < T : S_n = x\}$ be the number of visits to $x \in \mathbb{Z}$ before the return time T. Show that $\mathbb{E}(N_T(x)) = 1$ for *all* $x \in \mathbb{Z}$.

Hint: Establish a property of the function $f(x) = \mathbb{E}(N_T(x))$ that will imply this. It may help to consider simple random walk first. The argument has to include a justification why $\mathbb{E}(N_T(x)) < \infty$. (5pts)

(d) Show that $\mathbb{E}(T) = +\infty$. Hint: Use (c). (3pts)

Problem 3: Let (X, d) be a complete separable metric space. The purpose of this problem is to show that X is homeomorphic to a Borel subset of a compact metric space. This allows us to apply probabilistic methods (such as Kolmogorov's consistency theorem or the construction of Markov chains) to random variables that take values in such a space X.

For the proof we consider the set $Z = [0,1]^{\mathbb{N}} = \{u \colon \mathbb{N} \to [0,1]\}$ and two metrics ρ and σ on Z defined as

$$\rho(u,v) := \sup_{n\in\mathbb{N}} |u(n) - v(n)| \quad \text{and} \quad \sigma(u,v) := \sum_{n=1}^{\infty} \frac{1}{2^n} |u(n) - v(n)|$$

for $u, v \in Z$.

(a) Show that the metric space (Z, σ) is compact. (2pts)

We will assume without loss of generality that (X, d) has diameter bounded by 1 (if this is not true, then we replace the original metric d on X by the topologically equivalent metric $\tilde{d} = d \wedge 1$). Let x_n, $n \in \mathbb{N}$, be a countable dense subset in X and define $F \colon X \to Z$ by assigning to $x \in X$ the element $F(x) \in Z$ defined as $F(x)(n) = d(x, x_n)$ for $n \in \mathbb{N}$. Let $S = F(X)$.

(b) Show that the map F is an isometry of (X, d) onto a (S, ρ) and that S is closed in (Z, ρ). (2pts)

(c) Show that F is a homeomorphism of (X, d) onto (S, σ). (2pts)

(d) For $k \in \mathbb{N}$ let V_k be the set of all points $u \in Z$ for which there exists a set $N \subseteq Z$ such that $u \in N$, N is open in (Z, σ), $N \cap S \neq \emptyset$, $\operatorname{diam}_\sigma(N) < 1/k$, and $\operatorname{diam}_\rho(N \cap S) < 1/k$ (here "diam" refers to the diameter of a set and the subscript to the underlying metric).

Show that V_k is open in (Z, σ) and that $S \subseteq V_k$ for each $k \in \mathbb{N}$. (3pts)

(e) Show that $S = \bigcap_{k\in\mathbb{N}} V_k$ and conclude that X is homeomorphic to a Borel subset of a compact metric space. (3pts)

Problem 4: Let Z_n, $n \in \mathbb{N}$, be independent random variables with a standard normal distribution defined on some underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let f_n, $n \in \mathbb{N}$, be an arbitrary Hilbert space basis of $L^2([0,1])$ and define

$$g_n(t) = \int_0^t f_n(u)\, du$$

for $n \in \mathbb{N}$ and $t \in [0,1]$. Note that each function g_n is continuous on $[0,1]$. Now set

$$B_t(\omega) = \sum_{n=1}^{\infty} Z_n(\omega) g_n(t) \tag{1}$$

for $\omega \in \Omega$ and $t \in [0,1]$. The purpose of this problem is to show that for almost every $\omega \in \Omega$ the series in (1) converges uniformly for $t \in [0,1]$. Then the process

B_t, $t \in [0,1]$, has continuous sample paths almost surely. By the discussion in class this implies that B_t is a version of Brownian motion on $[0,1]$.

(a) Let $a = \{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers with

$$\|a\|_2 := \left(\sum_{n=1}^{\infty} a_n^2\right)^{1/2} < \infty.$$

Define $X_n := \sum_{k=1}^{n} a_k Z_k$ for $n \in \mathbb{N}$ and $X_* = \sup_{n\in\mathbb{N}} |X_n|$. Show that then there exists a constant $C \geq 0$ independent of a such that $\mathbb{E}(X_*^4) \leq C\|a\|_2^4$. (3pts)

(b) For $n \in \mathbb{N}$ we denote by $\mathcal{D}_n$ the set of dyadic intervals I of the form

$$I = [(i-1)/2^n, i/2^n] \quad \text{for } i = 1, \dots, 2^n.$$

If h is an arbitary function on $[0,1]$ and $I = [(i-1)/2^n, i/2^n]$, we write

$$\Delta(h, I) := h(i/2^n) - h((i-1)/2^n).$$

Show that if $h\colon [0,1] \to \mathbb{R}$ is a continuous function with $h(0) = 0$, then

$$\sup_{t\in[0,1]} |h(t)| \leq \sum_{n=1}^{\infty} \sup_{I\in\mathcal{D}_n} |\Delta(h, I)|.$$

(3pts)

(c) Let $N \in \mathbb{N}$ and $I \subseteq [0,1]$ be a dyadic interval. Then for $t \in [0,1]$ and $\omega \in \Omega$ we define

$$S_N(t,\omega) := \sum_{i=1}^{N} Z_i(\omega) g_i(t),$$

$$\Delta_N(I)(\omega) := \Delta(S_N(\cdot,\omega), I) \quad \text{and} \quad \Delta_N^*(I)(\omega) := \sup_{n\geq N} |\Delta_n(I)(\omega) - \Delta_N(I)(\omega)|.$$

Show that for each $n \geq N$,

$$\sup_{t\in[0,1]} |S_n(t,\omega) - S_N(t,\omega)| \leq \sum_{i=1}^{\infty} \sup_{I\in\mathcal{D}_i} \Delta_N^*(I)(\omega) =: \epsilon_N(\omega).$$

(2pts)

(d) Use (a) to find a good estimate for $\mathbb{E}[\Delta_N^*(I)^4]$ and use this to show that for the random variable ϵ_N defined in (c) we have $\mathbb{E}(\epsilon_N) \to 0$ as $N \to \infty$. (4pts)

(e) Show that for almost every $\omega \in \Omega$ we have

$$\sup_{t\in[0,1]} |S_k(t,\omega) - S_n(t,\omega)| \to 0$$

as $k, n \to \infty$ and conclude that the series in (1) converges uniformly. (2pts)