Final Exam (due: Mo, 6/15, 12pm)

Problem 1: Suppose the process X_t, $t \geq 0$, is a martingale adapted to a right-continuous filtration $\mathcal{F}_t$ on an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and suppose that it has right-continuous sample paths. Let T be a bounded stopping time. Then there exists $N \in \mathbb{N}_0$ such that $T \leq N$.

(a) For $n \in \mathbb{N}_0$ we consider the discretization T_n of T defined as $T_n = k/2^n$ whenever $T \in [(k-1)/2^n, k/2^n)$ for some $k \in \mathbb{N}$. We may assume that $T_n \leq N$ for each $n \in \mathbb{N}_0$. Show that then $\mathbb{E}(X_N|\mathcal{F}_{T_n}) = X_{T_n}$ a.s. for $n \in \mathbb{N}_0$ and that the family of random variables $\{X_{T_n} : n \in \mathbb{N}_0\}$ is uniformly integrable. (3pts)

(b) Show that $\mathbb{E}(X_N|\mathcal{F}_T) = X_T$. (3pts)

(c) Let S be another stopping time and assume that $S \leq T$. Show that then $\mathbb{E}(X_T|\mathcal{F}_S) = X_S$ a.s. and $\mathbb{E}(X_S) = \mathbb{E}(X_T)$. (4pts)

Problem 2: We consider 1-dimensional Brownian motion B_t starting at 0. Let $a > 0$ and define

$$T = \inf\{t \geq 0 : B_t \notin (-a, a)\}.$$

(a) Justify why T is an almost surely finite stopping time. (2pts)

(b) Show that for each $\lambda > 0$ we have

$$\mathbb{E}_0(e^{-\lambda T}) = 1/\cosh(a\sqrt{2\lambda}).$$

Hint: Apply Problem 1 to a suitable exponential martingale. (4pts)

(c) Use (b) to show that $\mathbb{E}_0(T^n) < \infty$ for all $n \in \mathbb{N}$ and to compute $\mathbb{E}_0(T)$ and $\mathbb{E}_0(T^2)$. (4pts)

Problem 3: (a) Suppose the process X_t, $t \geq 0$, is a non-negative submartingale adapted to a right-continuous filtration $\mathcal{F}_t$ on an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and suppose that it has continuous sample paths. Define

$$X_t^* = \sup_{0 \leq s \leq t} X_t$$

for $t \geq 0$. Show that then $\mathbb{P}(X_t^* \geq \lambda) \leq \mathbb{E}(X_t)/\lambda$ for $\lambda > 0$ and $t \geq 0$. Hint: Reduce to the discrete-time case. (2pts)

For the rest of this problem we consider 1-dimensional Brownian motion B_t starting at 0 with the usual right-continuous filtration $\mathcal{F}_t$.

(b) Define

$$B_t^* = \sup_{0 \leq s \leq t} |B_t|$$

for $t \geq 0$. Show that then $\mathbb{P}_0(B_t^* \geq \lambda) \leq 2e^{-\lambda^2/2t}$ for $\lambda > 0$ and $t > 0$. (2pts)

(c) Show that for each $\epsilon \in (0,1)$ there exists $\delta > 0$ with the following property: if we define the event

$$A_t = \{|B_{t+s} - B_t| < \epsilon \text{ for all } s \in [0, \delta]\},$$

then

$$\mathbb{P}_0(A_t | \mathcal{F}_t) \geq 1 - \epsilon \quad \text{a.s.}$$

for each $t \geq 0$. (3pts)

(d) Show that with positive probability B_t follows a given continuous function f with $f(0) = 0$ on a finite time interval arbitrarily closely. More precisely, let $t_0 \geq 0$, $\epsilon > 0$, and $f \colon \mathbb{R} \to \mathbb{R}$ be a continuous function with $f(0) = 0$. Show that then

$$\mathbb{P}_0(|B_t - f(t)| < \epsilon \text{ for each } t \in [0, t_0]) > 0.$$

(3pts)

Problem 4: (a) Let X, Y, Z be random variables taking values in some measurable space $\mathcal{X}$. Show that if X and Y are independent, X and Z are independent, and Y and Z have the same distribution, then the random variables (X, Y) and (X, Z) (taking values in $\mathcal{X}^2$) also have the same distribution. (2pts)

(b) Let X_t and Y_t, $t \geq 0$, be stochastic processes with values in $\mathbb{R}$ adapted to a right-continuous filtration $\mathcal{F}_t$ on an underlying probability space. Let T be a stopping time and suppose that $Y_0 = 0$ and that the processes X_t and Y_t have continuous sample paths. We consider two processes Z_t^+ and Z_t^- defined as

$$Z_t^{\pm} = \begin{cases} X_t & \text{for } 0 \leq t \leq T, \\ X_T \pm Y_{t-T} & \text{for } t > T. \end{cases}$$

Show that Z_t^+ and Z_t^- are also adapted processes with continuous sample paths. (2pts)

(c) We use the notation and assumptions as in (b), and consider the stochastic processes in (b) as random variables taking values in $\mathcal{X} = C([0,\infty), \mathbb{R})$. Suppose that the processes X_t and Y_t are independent, and that the processes Y_t and $-Y_t$ have the same distribution. Show that then the processes Z_t^+ and Z_t^- also have the same distribution. (2pts)

For the rest of this problem we consider 1-dimensional Brownian motion B_t starting at 0 with the usual right-continuous filtration $\mathcal{F}_t$.

(d) Establish the *reflection principle* for Brownian motion: Let T is a stopping time with $\mathbb{P}_x(T = +\infty) = 0$ for each $x \in \mathbb{R}$, and define a new process as

$$\widetilde{B}_t = \begin{cases} B_t & \text{for } 0 \leq t \leq T, \\ 2B_T - B_t & \text{for } t > T. \end{cases}$$

Show that then the process $\widetilde{B}_t$ is also a Brownian motion starting at 0. (4pts)

Problem 5: We consider the space $\mathcal{X} = C([0,\infty),\mathbb{R})$ equipped with the topology of locally uniform convergence. We denote the Borel σ-algebra on $\mathcal{X}$ by $\mathscr{B}$. If $F \subseteq [0,\infty)$ is a finite set, we let $\pi_F \colon \mathcal{X} \to \mathbb{R}^F$ be the projection map given by $f \in \mathcal{X} \mapsto \pi_F(f) := f|F \in \mathbb{R}^F$.

Suppose μ and μ_n for $n \in \mathbb{N}$ are probability measures on $(\mathcal{X},\mathscr{B})$. As usual, we say that μ_n *converges weakly* to μ (written $\mu_n \to \mu$ weakly) if $\int G\,d\mu_n \to \int G\,d\mu$ as $n \to \infty$ for all bounded continuous functions $G \colon \mathcal{X} \to \mathbb{R}$.

Show that $\mu_n \to \mu$ weakly if and only if the family μ_n, $n \in \mathbb{N}$, of measures is tight (i.e., for each $\epsilon > 0$ there exists a compact set $K \subseteq \mathcal{X}$ such that $\mu_n(K) > 1-\epsilon$ for all $n \in \mathbb{N}$) and we have weak convergence of all finite-dimensional marginals (i.e., $(\pi_F)_*(\mu_n) \to (\pi_F)_*(\mu)$ weakly for each finite set $F \subseteq [0,\infty)$). (10pts)