**Final** (due: Mo, 3/23)

**Problem 1:** Let $X_n$ for $n \in \mathbb{N}$ be i.i.d. random variables with $\mathbb{E}(X_n) = 0$ and $\mathbb{E}(X_n^2) = 1$. For $n \in \mathbb{N}$ and $t > 0$ define

$$S_n(t) = \frac{1}{\sqrt{n}}(X_1 + \cdots + X_{\lfloor nt \rfloor}).$$

Here $\lfloor x \rfloor = \max\{k \in \mathbb{Z} : k \leq x\}$ for $x \in \mathbb{R}$.

(a) Show that for fixed $t > 0$ we have

$$S_n(t) \Rightarrow B(t) \text{ as } n \to \infty,$$

where $B(t) \sim \mathcal{N}(0, t)$. (3pts)

(b) Prove the following generalization of (a): Fix $0 \leq t_1 < \cdots < t_d$ and define an $\mathbb{R}^d$-valued random variable as

$$Y_n = (S_n(t_1), \ldots, S_n(t_d)) \text{ for } n \in \mathbb{N}.$$

Show that then

$$Y_n \Rightarrow B \text{ as } n \to \infty,$$

where $B$ is a Gaussian. Find the expectation and the covariance matrix of $B$. (7pts)

**Problem 2:** Let $X_n$, $n \in \mathbb{N}_0$, be a martingale with $\mathbb{E}(X_n^2) < \infty$ for $n \in \mathbb{N}_0$.

(a) Show that

$$\mathbb{E}[(X_l - X_k)(X_n - X_m)] = 0,$$

whenever $0 \leq k \leq l \leq m \leq n$. (4pts)

(b) Show that

$$\sum_{n=1}^{\infty} \mathbb{E}[(X_n - X_{n-1})^2] < \infty$$

if and only if there exists a random variable $X_\infty$ with $\mathbb{E}(X_\infty^2) < \infty$ such that $X_n \to X_\infty$ a.s. and in $\mathscr{L}^2$. (6pts)

**Problem 3:** Let $\xi_n$ for $n \in \mathbb{N}$ be i.i.d. random variables with

$$\mathbb{P}(\xi_n = 1) = \mathbb{P}(\xi_n = -1) = 1/2.$$

Define $X_0 = 0$ and $X_n = \xi_1 + \cdots + \xi_n$ for $n \in \mathbb{N}$. For fixed $a, b \in \mathbb{N}$ we consider the stopping times

$$T_a = \inf\{n \in \mathbb{N}_0 : X_n = a\}, \quad T_{-b} = \inf\{n \in \mathbb{N}_0 : X_n = -b\},$$

and $T = T_a \wedge T_{-b}$ (with respect to the natural filtration $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$, $n \in \mathbb{N}_0$).

(a) Compute $\mathbb{P}(T_a < T_{-b})$ and $\mathbb{E}(T)$. Since this was already discussed in class (with different notation), it is enough to give a brief outline for the justification. (2pts)

(b) Show that there exists an adapted process $Y_n$ such that

$$M_n = X_n^3 + Y_n, \ n \in \mathbb{N}_0,$$

is a martingale. (4pts)

(c) Compute $\mathbb{E}(T_a \,|\, T_a < T_{-b})$. (4pts)

**Problem 4:** As in the previous problem, let $\xi_n$ for $n \in \mathbb{N}$ be i.i.d. random variables with

$$\mathbb{P}(\xi_n = 1) = \mathbb{P}(\xi_n = -1) = 1/2,$$

and define $X_0 = 0$ and $X_n = \xi_1 + \cdots + \xi_n$ for $n \in \mathbb{N}$.

(a) Show that

$$\mathbb{P}(\sup_{0 \le k \le n} |X_k| \ge a) \le 2e^{-a^2/(2n)}$$

for all $a \ge 0$ and $n \in \mathbb{N}$. Hint: Consider a suitable exponential submartingale and use the inequality $\cosh(x) \le e^{x^2/2}$ for $x \ge 0$. (5pts)

(b) Show that

$$\limsup_{n\to\infty} \frac{|X_n|}{\sqrt{2n \log\log n}} \le 1 \text{ a.s.}$$

Hint: It suffices to show that

$$\mathbb{P}(|X_n| \ge (1+\epsilon)\sqrt{2n \log\log n} \text{ i.o.}) = 0.$$

for each $\epsilon > 0$. Sparsify! (5pts)

**Problem 5:** Recall that a graph $G$ consists of a (countable) set of vertices $V$ where some pairs of distinct vertices are connected by one undirected edge. We write $u \sim v$ if two distinct vertices $u, v \in V$ are joined by an edge and call the vertices *neighbors* in this case. We assume that each $v \in V$ has a finite and non-zero number of neighbors denoted by $d_v \in \mathbb{N}$.

A *simple random walk* on $G$ is a sequence $X_n$, $n \in \mathbb{N}_0$, of $V$-valued random variables, where $X_0 = v_0$ for a fixed $v_0 \in V$ (the starting point of the random walk); moreover, if $X_n = v_n$, then $X_{n+1}$ is one of the neighbors of $v_n$ chosen uniformly at random. (It is not hard to give a careful definition of the underlying probability space based Kolmogorov's consistency theorem).

We call the simple random walk on $G$ *recurrent* if for some choice of $X_0 = v_0$ the random walk almost surely visits each vertex infinitely often. Finally, we call a function $f\colon V \to \mathbb{R}$ *superharmonic* if

$$f(v) \ge \frac{1}{d_v} \sum_{u \sim v} f(u)$$

for each $v \in V$.

(a) Show that if $f\colon V \to \mathbb{R}$ is superharmonic and $X_n$, $n \in \mathbb{N}_0$, is a simple random walk on $G$, then $M_n = f(X_n)$ is a supermartingale (with respect to the natural filtration induced by $X_n$). (4pts)

(b) Suppose that simple random walk on a graph $G$ is recurrent. Show that then every non-negative superharmonic function $f\colon V \to \mathbb{R}$ is constant. (6pts)