Homework 5 (due: Fr, 2/6)

Problem 1: Let $N$ and $X_n$ for $n \in \mathbb{N}$ be independent random variables. Suppose $N$ takes values in $\mathbb{N}_0$ and each $X_n$ has a Bernoulli distribution with parameter $p \in (0, 1)$, i.e., $\mathbb{P}(X_n = 1) = p$ and $\mathbb{P}(X_n = 0) = 1 - p$. Define two random variables $Y$ and $Z$ as

$$Y = \#\{n \in \mathbb{N} : X_n = 1 \text{ and } n \leq N\}, \quad Z = \#\{n \in \mathbb{N} : X_n = 0 \text{ and } n \leq N\}.$$  

Then $N = Y + Z$ and $Y$ and $Z$ are obtained by “thinning” $N$.

Show that if $Y$ and $Z$ are independent, then $N$ has a Poisson distribution.

Problem 2: Let $X$ be a random variable, and $\mathcal{A}$ and $\mathcal{B}$ be $\sigma$-algebras contained in the $\sigma$-algebra of the underlying probability space. Show that if $\mathcal{B}$ and $\sigma(\sigma(X), \mathcal{A})$ and independent, then

$$\mathbb{E}(X|\sigma(\mathcal{A}, \mathcal{B})) = \mathbb{E}(X|\mathcal{A}).$$

Hint: Apply Dynkin’s $\pi$-$\lambda$-theorem.