Homework 3 (due: Fr, 1/23)

Problem 1: Recall that a stochastic process B_t, $t \in [0, \infty)$, is called a *Brownian motion* if it is a Lévy process with almost surely continuous sample paths and $B_t \sim \mathcal{N}(0, t)$ for $t \geq 0$. Show that these conditions uniquely determine the marginals of the process; more precisely, show that if $n \in \mathbb{N}$, $0 \leq t_1 < \cdots < t_n$, and $X = (B_{t_1}, \ldots, B_{t_n})$, then X is an $\mathbb{R}^n$-valued Gaussian random variable and one can explicitly compute the expectation μ and covariance matrix C of X.

Problem 2: Let N_n for $n \in \mathbb{N}$ be i.i.d. random variables with $N_n \sim \mathcal{N}(0, 1)$ on some underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

(a) Suppose $\{a_n\}_{n \in \mathbb{N}}$ is a sequence of real numbers with $\sum_{n=1}^{\infty} a_n^2 < \infty$. Show that then

$$Z = \sum_{n=1}^{\infty} a_n N_n$$

converges in $L^2(\Omega, \mathbb{P})$ and that Z is a Gaussian random variable. Compute the expectation and the variance of Z.

(b) Let $\{\varphi_n\}$ be a Hilbert space basis of $L^2([0, 1])$ and define

$$\psi_n(t) = \int_0^t \varphi_n(t)\, dt \quad \text{for } n \in \mathbb{N} \text{ and } t \in [0, 1].$$

Let

$$B_t := \sum_{n=1}^{\infty} \psi_n(t) N_n \quad \text{for } t \in [0, 1].$$

Show that B_t is a Gaussian random variable and that $\mathbb{E}(B_s B_t) = \min\{s, t\}$ for $s, t \in [0, 1]$.