Midterm (due: Mo, 3/9)

Problem 1: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

(a) Suppose that X and Y are two random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with finite expectations. Show that

$$X \leq Y \text{ a.s. if and only if } \mathbb{E}(X; A) \leq \mathbb{E}(X; A) \text{ for all } A \in \mathcal{F}.$$

(3pts)

(b) Let $\mathcal{B} \subseteq \mathcal{F}$ be a σ-algebra and $A \in \mathcal{F}$. Define the *conditional probability of A given $\mathcal{B}$* as

$$\mathbb{P}(A \,|\, \mathcal{B}) := \mathbb{E}(\mathbb{1}_A \,|\, \mathcal{B})$$

(note that this is a random variable!). Show that for some $\epsilon > 0$ we have $\mathbb{P}(A \,|\, \mathcal{B}) \geq \epsilon$ a.s. if and only if $\mathbb{P}(A \,|\, B) \geq \epsilon$ for all $B \in \mathcal{B}$ with $\mathbb{P}(B) > 0$. (3pts)

(c) Let T be a stopping time with respect to a filtration $\mathcal{F}_n$, $n \in \mathbb{N}_0$, on $(\Omega, \mathcal{F}, \mathbb{P})$. Show that if there exists $N \in \mathbb{N}_0$ and $\epsilon > 0$ such that

$$\mathbb{P}(T \leq n + N \,|\, \mathcal{F}_n) \geq \epsilon \text{ a.s. for all } n \in \mathbb{N}_0,$$

then $\mathbb{E}(T) < \infty$. (4pts)

Problem 2: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathcal{A} \subseteq \mathcal{F}$ be a σ-algebra, and X be a random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}(|X|) < \infty$.

Show that if X and $Y := \mathbb{E}(X \,|\, \mathcal{A})$ have the same law (i.e., $\nu_X = \nu_Y$), then $X = Y$ a.s. Hint: First show that X and Y have the same sign a.s. Apply this to $X - c$ for $c \in \mathbb{R}$. (10pts)

Problem 3: Let X_n, $n \in \mathbb{N}_0$, be a martingale with respect to a given filtration $\mathcal{F}_n$, $n \in \mathbb{N}_0$, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and T be a stopping time. Suppose that $\mathbb{E}(T) < \infty$ and that there exists $M \geq 0$ such that

$$|X_{n+1} - X_n| \leq M \text{ for all } n \in \mathbb{N}_0.$$

Show that then X_T has finite expectation and

$$\mathbb{E}(X_T) = \mathbb{E}(X_0).$$

Hint: Consider the stopped process X_n^T and pass to the limit. (10pts)

Problem 4: Let ξ_n, $n \in \mathbb{N}_0$, be i.i.d. random variables with $\mathbb{P}(\xi_n = 1) = p \in (1/2, 1)$ and $\mathbb{P}(\xi_n = -1) = q := 1 - p \in (0, 1/2)$. Define $X_0 = 0$ and

$$X_n = \xi_1 + \cdots + \xi_n$$

for $n \in \mathbb{N}$. Note that X_n models the winnings of a player in a coin-flipping game where the coin is biased in favor of the player.

(a) Show that $\mathbb{P}(\lim_{n\to\infty} X_n = +\infty) = 1$. (2pts)

(b) Show that there exists a number $\alpha \neq 0$ such that $S_n := \exp(\alpha X_n)$, $n \in \mathbb{N}_0$, is a martingale with respect to the natural filtration $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$, $n \in \mathbb{N}_0$. (3pts)

(c) Let $a \in \mathbb{N}$. The player goes bankrupt if $X_n = -a$ for some $n \in \mathbb{N}$. Compute the probability for this event. (5pts)

Problem 5: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a given filtration $\mathcal{F}_n$, $n \in \mathbb{N}_0$. Below all (sub-)martingales are with respect to this filtration.

(a) Suppose Z_n, $n \in \mathbb{N}_0$, is a non-negative submartingale that is bounded in L^1. Show that then for each $n \in \mathbb{N}_0$ the limit $Y_n = \lim_{k\to\infty} \mathbb{E}(Z_k \mid \mathcal{F}_n)$ exists a.s. and has finite expectation. (5pts)

(b) Suppose X_n, $n \in \mathbb{N}_0$, is a martingale that is bounded in L^1. Show that then there exist non-negative martingales U_n and V_n such that $X_n = U_n - V_n$ for $n \in \mathbb{N}_0$. (5pts)