Final (due: Mo, 3/23)

Problem 1: Let $X_n$ for $n \in \mathbb{N}$ be i.i.d. random variables with $\mathbb{E}(X_n) = 0$ and $\mathbb{E}(X_n^2) = 1$. For $n \in \mathbb{N}$ and $t > 0$ define
$$S_n(t) = \frac{1}{\sqrt{n}}(X_1 + \cdots + X_{\lfloor nt \rfloor}).$$
Here $\lfloor x \rfloor = \max\{k \in \mathbb{Z} : k \leq x \}$ for $x \in \mathbb{R}$.

(a) Show that for fixed $t > 0$ we have
$$S_n(t) \Rightarrow B(t) \text{ as } n \to \infty,$$
where $B(t) \sim \mathcal{N}(0, t)$.

(b) Prove the following generalization of (a): Fix $0 \leq t_1 < \cdots < t_d$ and define an $\mathbb{R}^d$-valued random variable as
$$Y_n = (S_n(t_1), \ldots, S_n(t_d)) \text{ for } n \in \mathbb{N}.$$
Show that then
$$Y_n \Rightarrow B \text{ as } n \to \infty,$$
where $B$ is a Gaussian. Find the expectation and the covariance matrix of $B$.

(7pts)

Problem 2: Let $X_n$, $n \in \mathbb{N}_0$, be a martingale with $\mathbb{E}(X_n^2) < \infty$ for $n \in \mathbb{N}_0$.

(a) Show that
$$\mathbb{E}[(X_l - X_k)(X_n - X_m)] = 0,$$
whenever $0 \leq k \leq l \leq m \leq n$.

(b) Show that
$$\sum_{n=1}^{\infty} \mathbb{E}[(X_n - X_{n-1})^2] < \infty$$
if and only if there exists a random variable $X_\infty$ with $\mathbb{E}(X_\infty^2) < \infty$ such that $X_n \to X_\infty$ a.s. and in $L^2$.

(6pts)

Problem 3: Let $\xi_n$ for $n \in \mathbb{N}$ be i.i.d. random variables with
$$\mathbb{P}(\xi_n = 1) = \mathbb{P}(\xi_n = -1) = 1/2.$$
Define $X_0 = 0$ and $X_n = \xi_1 + \cdots + \xi_n$ for $n \in \mathbb{N}$. For fixed $a, b \in \mathbb{N}$ we consider the stopping times
$$T_a = \inf\{n \in \mathbb{N}_0 : X_n = a\}, \quad T_{-b} = \inf\{n \in \mathbb{N}_0 : X_n = -b\},$$
and $T = T_a \wedge T_{-b}$ (with respect to the natural filtration $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$, $n \in \mathbb{N}_0$).
(a) Compute \( \mathbb{P}(T_a < T_b) \) and \( \mathbb{E}(T) \). Since this was already discussed in class (with different notation), it is enough to give a brief outline for the justification. (2pts)

(b) Show that there exists an adapted process \( Y_n \) such that
\[
M_n = X_n^3 + Y_n, \quad n \in \mathbb{N}_0,
\]
is a martingale. (4pts)

(c) Compute \( \mathbb{E}(T_a | T_a < T_b) \). (4pts)

**Problem 4:** As in the previous problem, let \( \xi_n \) for \( n \in \mathbb{N} \) be i.i.d. random variables with
\[
\mathbb{P}(\xi_n = 1) = \mathbb{P}(\xi_n = -1) = 1/2,
\]
and define \( X_0 = 0 \) and \( X_n = \xi_1 + \cdots + \xi_n \) for \( n \in \mathbb{N} \).

(a) Show that
\[
\mathbb{P}(\sup_{0 \leq k \leq n} |X_k| \geq a) \leq 2e^{-a^2/(2n)}
\]
for all \( a \geq 0 \) and \( n \in \mathbb{N} \). Hint: Consider a suitable exponential submartingale and use the inequality \( \cosh(x) \leq e^{x^2/2} \) for \( x \geq 0 \). (5pts)

(b) Show that
\[
\limsup_{n \to \infty} \frac{|X_n|}{\sqrt{2n \log \log n}} \leq 1 \text{ a.s.}
\]
Hint: It suffices to show that
\[
\mathbb{P}(|X_n| \geq (1 + \epsilon) \sqrt{2n \log \log n} \text{ i.o.}) = 0.
\]
for each \( \epsilon > 0 \). Sparsify! (5pts)

**Problem 5:** Recall that a graph \( G \) consists of a (countable) set of vertices \( V \) where some pairs of distinct vertices are connected by one undirected edge. We write \( u \sim v \) if two distinct vertices \( u, v \in V \) are joined by an edge and call the vertices neighbors in this case. We assume that each \( v \in V \) has a finite and non-zero number of neighbors denoted by \( d_v \in \mathbb{N} \).

A simple random walk on \( G \) is a sequence \( X_n, n \in \mathbb{N}_0, \) of \( V \)-valued random variables, where \( X_0 = v_0 \) for a fixed \( v_0 \in V \) (the starting point of the random walk); moreover, if \( X_n = v_n \), then \( X_{n+1} \) is one of the neighbors of \( v_n \) chosen uniformly at random. (It is not hard to give a careful definition of the underlying probability space based Kolmogorov’s consistency theorem).

We call the simple random walk on \( G \) recurrent if for some choice of \( X_0 = v_0 \) the random walk almost surely visits each vertex infinitely often. Finally, we call a function \( f : V \to \mathbb{R} \) superharmonic if
\[
f(v) \geq \frac{1}{d_v} \sum_{u \sim v} f(u)
\]
for each $v \in V$.

(a) Show that if $f: V \to \mathbb{R}$ is superharmonic and $X_n, n \in \mathbb{N}_0$, is a simple random walk on $G$, then $M_n = f(X_n)$ is a supermartingale (with respect to the natural filtration induced by $X_n$). (4pts)

(b) Suppose that simple random walk on a graph $G$ is recurrent. Show that then every non-negative superharmonic function $f: V \to \mathbb{R}$ is constant. (6pts)