Math 275A, Probability Theory

Fall 2014

Midterm

Name:

There are five problems with a total of 50 points.

Problem 1: A random variable X has a *Cauchy distribution* if it has a probability density function given by

$$\rho(x) = \frac{1}{\pi(1+x^2)} \text{ for } x \in \mathbb{R}.$$

(a) Show that if X has a Cauchy distribution, then its characteristic function is given by

$$\varphi_X(t) = \mathbb{E}(e^{itX}) = e^{-|t|} \text{ for } t \in \mathbb{R}.$$

(2pts)

(b) Show that if $X_1, \ldots, X_n$ are independent random variables with a Cauchy distribution, then

$$S = (X_1 + \cdots + X_n)/n$$

also has a Cauchy distribution. (2pts)

(c) Suppose X_n, $n \in \mathbb{N}$, is a sequence of independent random variables, each with a Cauchy distribution. Show that there is no number $\mu \in \mathbb{R}$ such that we have convergence in probability

$$S_n = (X_1 + \cdots + X_n)/n \to \mu.$$

(2pts)

(d) Why does (c) not contradict the weak law of large numbers? (2pts)

Problem 2: (a) Suppose that X is a random variable such that for each Borel set $B \subseteq \mathbb{R}$ we have $\mathbb{P}(X \in B) = 0$ or $\mathbb{P}(X \in B) = 1$. What can you say about X? (4 pts)

(b) Suppose X is a random variable such that X and λX have the same distribution for each $\lambda > 0$. What can you say about X? (4 pts)

Problem 3: Let X and X_n for $n \in \mathbb{N}$ be random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

(a) Formulate and prove a version of the following general principle: if, in a suitable sense, we have *fast enough* convergence $X_n \to X$ in probability, then $X_n \to X$ almost surely. (2pts)

(b) Your version of the principle in (a) should be good enough to prove the following fact: if $X_n \to X$ in probability, then $X_{n_k} \to X$ almost surely along a suitable subsequence. (2pts)

(c) Use Lévy's continuity theorem to show that if $X_n \to X$ almost surely, then then we have convergence $X_n \Rightarrow X$ in distribution. (2pts)

(d) Use the Portmanteau theorem to show that if $X_n \to X$ in probability, then $X_n \Rightarrow X$. (2pts)

(e) Give an alternative proof of the statement in (d) based on (b) and (c). (2pts)

Problem 4: Let X_n for $n \in \mathbb{N}$ be independent random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For $n \geq 1$ and $m \geq n$ we consider the σ-algebras $\mathcal{A}_{n,m} = \sigma(X_n, \ldots, X_m)$ and $\mathcal{A}_n = \sigma(X_n, X_{n+1}, \ldots)$. So, for example, $\mathcal{A}_n$ is generated by $X_n, X_{n+1}, \ldots$ and the smallest σ-algebra on Ω that contains all events $\{X_k \in B\}$, where $k \geq n$ and $B \subseteq \mathbb{R}$ is a Borel set.

(a) Show that $\{X_k + \cdots + X_{k+l} \in B\} \in \mathcal{A}_n$, whenever $k \geq n$, $l \geq 0$, and $B \subseteq \mathbb{R}$ is a Borel set. (2pts)

(b) Consider the *tail* σ-algebra $\mathcal{T} = \bigcap_{n=1}^{\infty} \mathcal{A}_n$. Show that the event

$$C = \left\{ \omega \in \Omega : \sum_{n=1}^{\infty} X_n(\omega) \text{ converges} \right\}$$

belongs to $\mathcal{T}$. (2pts)

(c) Show that $\mathcal{A}_{1,n}$ and $\mathcal{A}_{k,l}$ are independent whenever $1 \leq n < k \leq l$. Hint: Consider suitable π-systems. (2pts)

(d) Show that $\mathcal{A}_{1,n}$ and $\mathcal{A}_k$ are independent whenever $1 \leq n < k$. (2pts)

(e) Show that $\mathcal{A}_{1,n}$ and $\mathcal{T}$ are independent for each $n \geq 1$. (2pts)

(f) Show that $\mathcal{A}_1$ and $\mathcal{T}$ are independent. (2pts)

(g) Show that for the event C defined in (b) we have $\mathbb{P}(C) = 0$ or $\mathbb{P}(C) = 1$. (2pts)

Problem 5: Let μ_n, $n \in \mathbb{N}$, be a sequence of probability measures on $\mathbb{R}^d$. Show that the sequence is tight if and only if there exists a constant $C \geq 0$, and a measurable function $f \geq 0$ with $f(x) \to +\infty$ as $|x| \to \infty$ such that

$$\int_{\mathbb{R}^d} f \, d\mu_n \leq C$$

for all $n \in \mathbb{N}$. (10 pts)