Math 275A, Probability Theory

Fall 2014

Final Exam

Name:

There are five problems with a total of 50 points.

Problem 1: Suppose X_n for $n \in \mathbb{N}$ are i.i.d. random variables with $\mathbb{E}(X_n) = 0$ and $\mathbb{E}(X_n^2) = \sigma^2 \in [0, \infty)$. Let $S_n = X_1 + \cdots + X_n$ for $n \in \mathbb{N}$. Show that then

$$\lim_{n \to \infty} \mathbb{E}(|S_n|/\sqrt{n}) = \sigma\sqrt{2/\pi}.$$

(10pts)

Problem 2: A *simple random walk* on $\mathbb{Z}$ starts at 0 and at each time $n \in \mathbb{N}_0$ goes one step to the left or right with equal probability. So if X_n for $n \in \mathbb{N}$ are i.i.d. random variables with $\mathbb{P}(X_n = 1) = \mathbb{P}(X_n = -1) = 1/2$, and $S_n = X_1 + \cdots + X_n$, then S_n is the position of the random walk at time n (before the next step is taken at time n).

(a) Show that $\limsup_{n\to\infty} S_n/\sqrt{n} = +\infty$ almost surely. Hint: Use Kolmogorov's 0-1-law (essentially proved in Midterm, Prob. 4). (5 pts)

(b) Use (a) to show that the simple random walk on $\mathbb{Z}$ is *recurrent*, i.e.,

$$\mathbb{P}(S_n = 0 \text{ i.o.}) = 1,$$

and so the random walk almost surely returns to 0 infinitely often. (5 pts)

Problem 3: (a) Let X_n and Y_n for $n \in \mathbb{N}$ be random variables. Show that if $X_n \Rightarrow X$ and $Y_n \to a$ almost surely, where X is a random variable and $a \in \mathbb{R}$ a constant, then $X_n Y_n \Rightarrow aX$. (5pts)

(b) Let X_n for $n \in \mathbb{N}$ be i.i.d. random variables with $X_n \geq 0$, finite second moments, $\mathbb{E}(X_n) = 1$, and $\mathrm{Var}(X_n) = \sigma^2 \in (0, \infty)$. Define $S_n = X_1 + \cdots + X_n$ for $n \in \mathbb{N}$. Show that then

$$2(\sqrt{S_n} - \sqrt{n}) \Rightarrow \sigma N,$$

where N is a standard normal random variable.

(5pts)

Problem 4: An $\mathbb{R}^n$-valued random variable X is called a *generalized Gaussian* if its characteristic function has the form

$$\varphi_X(u) = \mathbb{E}(e^{i(u\cdot X)}) = \exp(i(u\cdot\mu) - u^t M u), \quad u \in \mathbb{R}^n. \tag{1}$$

Here $\mu \in \mathbb{R}^n$, M is a real $(n \times n)$-matrix, and u^t denotes the transpose of the column vector $u \in \mathbb{R}^n$.

(a) If X is a generalized Gaussian, then in the representation (1) we may assume that M is symmetric (why?). Show that M is actually positive semi-definite, i.e., $u^t M u \geq 0$ for all $u \in \mathbb{R}^n$. (2pts)

(b) Show that if X is an $\mathbb{R}^n$-valued generalized Gaussian and A is a real $(k \times n)$-matrix, then $Y = AX$ is an $\mathbb{R}^k$-valued generalized Gaussian. (2pts)

(c) Suppose that X and X_k for $k \in \mathbb{N}$ are $\mathbb{R}^n$-valued random variables with $X_k \Rightarrow X$. Show that if X_k is a generalized Gaussian for each $k \in \mathbb{N}$, then X is also a generalized Gaussian. (6pts)

Problem 5: Let $X = C([0,1])$ be the vector space of all continuous real-valued functions on the interval $[0,1]$. Equipped with the supremum norm defined as

$$\|f\| = \sup_{t\in[0,1]} |f(t)|, \quad f \in X,$$

X is a Banach space, and in particular a topological space. Let $\mathcal{B}$ Borel σ-algebra on X.

(a) For each $t \in [0,1]$ we define a map $\pi_t\colon X \to \mathbb{R}$ by setting $\pi_t(f) = f(t)$ for $f \in X$; so π_t is the *evaluation functional* at time t. Show that $\mathcal{B}$ is the smallest σ-algebra on X containing all sets $\pi_t^{-1}(B)$, where $t \in [0,1]$ and $B \subseteq \mathbb{R}$ is a Borel set. (7pts)

(b) Similarly as in (a), for each finite set $I = \{t_1 < \cdots < t_n\} \subseteq [0,1]$ we can define a map $\pi_I\colon X \to \mathbb{R}^n$ be setting $\pi_I(f) = (f(t_1), \ldots, f(t_n)) \in \mathbb{R}^n$ for $f \in X$. Show that if μ is a probability measure on $(X, \mathcal{B})$, then it is uniquely determined by the family of measures consisting of all of its *marginals* $\mu_I := (\pi_I)_*\mu$, where $I \subseteq [0,1]$ is finite. (3pts)