# Conformal Invariant Processes in the Plane

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## 1 Koebe's distortion theorem

Notations:
$\mathbb{C}$ the complex plane,
$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the open unit disk,
$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ the Riemann sphere,
$\tilde{\mathbb{D}} = \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} = \{z \in \hat{\mathbb{C}} : |z| > 1\}$ the complement of the closed unit disk.

**Definition 1.1.** $\mathcal{S} = \{f : \mathbb{D} \to \mathbb{C} : f$ holomorphic and injective (conformal map onto its image), $f(0) = 0, f'(0) = 1\}$.

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \tag{1}$$

(Taylor series expansion).

$\Sigma = \{g : \tilde{\mathbb{D}} \to \hat{\mathbb{C}} : g$ holomorphic and injective (conformal map onto its image),

$$g(w) = w + b_0 + b_1/w + b_2/w^2 + \cdots \tag{2}$$

(Laurent series expansion at $\infty$)$\}$.
$g(\infty) = \infty, g'(w) = 1 + O(1/w^2)$ as $w \to \infty$, and $g'(\infty) = \lim_{w\to\infty} g'(w) = 1$.

Note. If $g$ is a holomorphic map on $\tilde{\mathbb{D}}$, $g(\infty) = \infty$, $g$ injective, then

$$g(1/z) = 1/z + b_0 + b_1 z + b_2 z^2 + \cdots$$

is holomorphic in $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$, and has $1^{\text{st}}$ order pole. The series in (2) converges uniformly on compact subsets in $\mathbb{C} \setminus \overline{\mathbb{D}}$.

**Theorem 1.2. (Area Theorem)** *If $g \in \Sigma$, then*

$$\text{Area}(\hat{\mathbb{C}} \setminus g(\tilde{\mathbb{D}})) = \pi\Big(1 - \sum_{n=1}^{\infty} n|b_n|^2\Big) \geq 0. \tag{3}$$

*In particular,*

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1 \quad \textit{and} \quad |b_1| \leq 1.$$

*Here, $|b_1| = 1$ iff*

$$g_\alpha(w) = w + b_0 + \frac{e^{2i\alpha}}{w}, \qquad \alpha \in \mathbb{R}.$$

*Proof.* Pick $r > 1$. Define $\gamma_r = g(re^{it})$, $t \in [0, 2\pi]$. $\gamma_r$ is a (parameterized) Jordan curve. The winding number

$$\operatorname{ind}_{\gamma_r}(w) = \begin{cases} 0, & \text{for} \quad w \in \operatorname{Out}(\gamma_r) \quad \text{(outside of } \gamma_r\text{)}, \\ \pm 1, & \phantom{\text{for}} \quad w \in \operatorname{In}(\gamma_r) \quad \text{(inside of } \gamma_r\text{)}. \end{cases}$$

By the Jordan curve theorem

$$\operatorname{In}(\gamma_r) = \hat{\mathbb{C}} \setminus g(\tilde{\mathbb{D}}) \cup g(\{w \in \mathbb{C} : 1 < |w| < r\}). \tag{4}$$

Moreover, $\operatorname{ind}_{\gamma_r}(u) = 1$ for $u \in \operatorname{In}(\gamma_r)$.

Figure 1: here

*Proof of* (4):

"$\supseteq$" part: the $\operatorname{ind}_{\gamma_r}(u) = 1$ follows from the homotopy invariance of the winding number (let $r \to +\infty$).

"$\subseteq$" part: it follows because every point on the right hand side is not on $\gamma_r$ or the set on the right hand side lies in the unbounded component of $\mathbb{C} \setminus \gamma_r$.

So

$$\hat{\mathbb{C}} \setminus g(\tilde{\mathbb{D}}) = \bigcap_{r>1} \operatorname{In}(\gamma_r),$$

and

$$\operatorname{Area}(\hat{\mathbb{C}} \setminus g(\tilde{\mathbb{D}})) = \lim_{r \to 1^+} \operatorname{Area}(\operatorname{In}(\gamma_r)).$$

By Green's theorem,

$$\frac{1}{2i} \int_{\gamma_r} \overline{u} du = \int_{\mathbb{C}} \operatorname{ind}_{\gamma_r}(u) dA(u) = \operatorname{Area}(\operatorname{In}(\gamma_r)),$$

where $dA(u)$ denotes the area differential. On the other hand,

$$\frac{1}{2i} \int_{\gamma_r} \overline{u} du = \frac{1}{2i} \int_0^{2\pi} \overline{g(re^{it})} g'(re^{it}) rie^{it} dt = \frac{1}{2} \int_0^{2\pi} \overline{g(w)} g'(w) w dt,$$

where it has been set $w = re^{it}$. From the Laurent series expansion (2)

$$g(w) = w + \sum_{n=0}^{\infty} \frac{b_n}{w^n}, \qquad g'(w) = 1 - \sum_{n=1}^{\infty} \frac{n b_n}{w^{n+1}}.$$

Note that $\overline{w} = r^2/w$ and

$$\int_0^{2\pi} w^k dw = \begin{cases} 0, & \text{for} \quad k \in \mathbb{Z} \setminus \{0\}, \\ 2\pi i, & \phantom{\text{for}} \quad k = 0. \end{cases}$$

By uniform convergence, we can integrate "term by term", and so

$$\begin{aligned}
\text{Area}(\text{In}(\gamma_r)) &= \frac{1}{2}\int_0^{2\pi} \overline{g(w)}g'(w)w dt \\
&= \frac{1}{2}\int_0^{2\pi} \Big(\overline{w} + \sum_{n=0}^{\infty} \frac{\overline{b}_n}{\overline{w}^n}\Big)\Big(w - \sum_{n=1}^{\infty} \frac{nb_n}{w^n}\Big) dt \\
&= \frac{1}{2}\int_0^{2\pi} \Big(|w|^2 - \sum_{n=1}^{\infty} n\frac{|b_n|^2}{|w|^{2n}}\Big) dt \\
&= \pi\Big(r^2 - \sum_{n=1}^{\infty} n|b_n|^2 r^{-2n}\Big) \\
&\to \pi\Big(1 - \sum_{n=1}^{\infty} n|b_n|^2\Big) \quad \text{as} \quad r \to 1 \qquad \text{(has to be justified).}
\end{aligned}$$

The first part follows!

So $|b_1| \le 1$. If $|b_1| = 1$, then $b_2 = b_3 = \cdots = 0$, and so

$$g(w) = g_\alpha(w) = w + b_0 + \frac{e^{2i\alpha}}{w}, \quad b_1 = e^{2i\alpha}, \quad \alpha \in \mathbb{R}.$$

(Joukovsky map)

Figure 2: $g_\alpha$, Joukovsky map

□

**Corollary 1.3.** *Let*

$$g(w) = w + b_0 + \frac{b_1}{w} + \frac{b_2}{w^2} + \cdots \in \Sigma.$$

*If $u \in \mathbb{C} \setminus g(\tilde{\mathbb{D}})$ (i.e. $u$ is omitted by $g$), then $|u - b_0| \le 2$ and if we have equality then $g$ is a Joukovsky map.*

*Proof.* Let

$$h(w) = \sqrt{g(w^2) - u} = w \cdot \sqrt{\frac{g(w^2)}{w^2} - \frac{u}{w^2}}$$

on $\tilde{\mathbb{D}}$. The function $\frac{g(w^2)}{w^2} - \frac{u}{w^2}$ is a zero-free holomorphic function on the simply connected domain $\tilde{\mathbb{D}}$. So $h$ is well defined. So

$$\begin{aligned}
h(w) &= w\Big(1 + \frac{b_0 - u}{w^2} + \cdots\Big)^{1/2} \\
&= w\Big(1 + \frac{1}{2}\frac{b_0 - u}{w^2} + \cdots\Big) = w + \frac{\tilde{b}_1}{w} + \cdots,
\end{aligned}$$

where $\tilde{b}_1 = \frac{1}{2}(b_0 - u)$. Note that $h$ is holomorphic and injective on $\tilde{\mathbb{D}}$. In fact, $h(w_1) = h(w_2) \Longrightarrow g(w_1^2) - u = g(w_2^2) - u \Longrightarrow w_1^2 = w_2^2 \Longrightarrow w_1 = \pm w_2$. If $w_1 = -w_2$, then $h(w_1) = h(w_2) = -h(w_1)$ ($h$ is odd), and so $h(w_1) = \infty$ (0 impossible!) $\Longrightarrow w_1 = w_2 = \infty$.

So $h \in \Sigma$. By Theorem 1.2, $|\frac{1}{2}(b_0 - u)| = |\tilde{b}_1| \leq 1$, equivalent to $|u - b_0| \leq 2$.

If $|u - b_0| = 2$, then $|\tilde{b}_1| = 1$, and so $h$ is a Joukovsky map, which implies that $g$ is a Joukovsky map:

$$h(w) = w + \frac{\tilde{b}_1}{w} = w + \frac{1}{2}\frac{b_0 - u}{w}.$$

$$gw^2 = h(w)^2 + u = w^2 + b_0 + \frac{1}{4}\frac{(b_0 - u)^2}{w^2}.$$

So

$$g(w) = w + b_0 + \frac{1}{4}\frac{(b_0 - u)^2}{w} = w + b_0 + \frac{\tilde{b}_1^2}{w}.$$

□

**Theorem 1.4.** *Let $f \in \mathcal{S}$.*

$$f(z) = z + a_2 z^2 + \cdots .$$

*Then*

i) $|a_2| \leq 2$,

ii) **(Koebe 1/4-Theorem)** *if $v \in \mathbb{C} \setminus f(\mathbb{D})$, then $|v| \geq 1/4$, i.e. $B(0, 1/4) \subseteq f(\mathbb{D})$.*

Figure 3: Koebe 1/4 – theorem

*We have equality in* i) *or* ii) *iff $f$ is a Koebe function, i.e.*

$$f(z) = e^{-i\alpha} K(e^{i\alpha}), \qquad K(z) = \frac{z}{(z-1)^2},$$

$$K(z) = z + 2z^2 + 3z^3 + \cdots .$$

Figure 4: Koebe function

**Remark.** A long-standing open problem was Bieberbach's conjecture: if $f \in \mathcal{S}$, then $|a_n| \leq n$ for $n \geq 2$, proved by de Brange (early 1980's).

*Proof.* If $f \in \mathcal{S}$, then $g(w) = 1/f(1/w) \in \Sigma$.

$$\begin{aligned} g(w) &= \frac{1}{1/w + a_2 2/w^2 + \cdots} = w \cdot \frac{1}{1 + a_2/w + a_3/w^2 + \cdots} \\ &= w\Big(1 - \Big(\frac{a_2}{w} + \frac{a_3}{w^2} + \cdots\Big) + \Big(\frac{a_2}{w} + \frac{a_3}{w^2} + \cdots\Big)^2 - \cdots\Big) \\ &= w\Big(1 - \frac{a_2}{w} + \frac{a_2^2 - a_3}{w^2} + \cdots\Big) \\ &= w - a_2 + \frac{a_2^2 - a_3}{w} + \cdots . \end{aligned}$$

Moreover, $u = 0$ is omitted by $g$!

i) By Corollary 1.3, $|a_2| = |0 - (-a_2)| (= |u - b_0|) \leq 2$.

If equality, then the proof of Corollary 1.3 shows

$$g(w) = w + b_0 + \frac{1}{4}\frac{(b_0 - u)^2}{w^2} = w - a_2 + \frac{1}{4}\frac{a_2^2}{w} = w\Big(1 - \frac{a_2}{2}\frac{1}{w}\Big)^2.$$

So

$$f(z) = \frac{1}{g(1/z)} = \frac{z}{(1 - (a_2/2)z)^2}, \qquad \text{where} \quad |a_2| = 2.$$

$f$ is the rotated Koebe function.

ii) If $v$ is omitted by $f$, then $u = 1/v$ is omitted by $g$. So by Corollary 1.3,

$$2 \geq |u - b_0| = \Big|\frac{1}{v} + a_2\Big|.$$

So

$$\Big|\frac{1}{v}\Big| \leq | - a_2| + \Big|\frac{1}{v} + a_2\Big| \leq 4,$$

equivalent to $|v| \geq 1/4$.

If $|v| = 1/4$, then $|1/v| = 4$ and $|a_2| = 2$. Again, $f$ is a rotation of the Koebe function. □

**Corollary 1.5.** *If $f \in \mathcal{S}$ and $\Omega = f(\mathbb{D})$, then*

$$\frac{1}{4} \leq \operatorname{dist}(0, \partial\Omega) \leq 1.$$

*Proof.* The first inequality follows from the 1/4 – Theorem. For the second inequality, let $d = \operatorname{dist}(0, \partial\Omega) < \infty$. Define $g(w) = f^{-1}(dw)$, $w \in \mathbb{D}$. Then $g(\mathbb{D}) \subseteq \mathbb{D}$, $g(0) = 0$; so by the Schwarz Lemma

$$1 \geq |g'(0)| = \frac{d}{|f'(0)|} = d.$$

□

**Lemma 1.6.** *If $f \in \mathcal{S}$, then*

$$\Big|(1 - |z|^2)\frac{f''(z)}{f'(z)} - 2\overline{z}\Big| \leq 4 \qquad \textit{for} \quad z \in \mathbb{D}.$$

*Proof.* Fix $z_0 \in \mathbb{D}$. Let $\varphi \in \operatorname{Aut}(\mathbb{D})$, $\varphi(0) = z_0$. Then

$$\varphi(z) = \frac{z + z_0}{1 + \overline{z}_0 z}, \quad \varphi'(z) = \frac{1 - |z_0|^2}{(1 + \overline{z_0} z)^2}, \quad \varphi''(z) = -2\frac{(1 - |z_0|)\overline{z_0}}{(1 + \overline{z_0} z)^3}.$$

Define $g = f \circ \varphi$. It is a conformal map on $\mathbb{D}$, but not normalized! Let

$$h = \frac{g - g(0)}{g'(0)}.$$

Then $h \in \mathcal{S}$ and $|a_2(h)| \leq 2$.

$$a_2(h) = \frac{1}{2}h''(0) = \frac{1}{2}\frac{g''(0)}{g'(0)}.$$

$$g' = (f' \circ \varphi) \cdot \varphi', \quad g'' = (f'' \circ \varphi) \cdot \varphi'^2 + (f' \circ \varphi) \cdot \varphi''.$$

$$g(0) = z_0, \quad g'(0) = f'(z_0)(1 - |z_0|^2),$$

$$g''(0) = f''(z_0)(1 - |z_0|^2)^2 + f'(z_0)(-2\overline{z_0}(1 - |z_0|^2)).$$

So

$$2 \geq |a_2(h)| = \frac{1}{2}\frac{|g''(0)|}{|g'(0)|} = \frac{1}{2}\left|\frac{f''(z_0)}{f'(z_0)}(1 - |z_0|^2) - 2\overline{z_0}\right|.$$

□

**Theorem 1.7. (Koebe's Distortion Theorem)** *Let $f \in \mathcal{S}$. Then for $z \in \mathbb{D}$*

i)
$$\frac{1-|z|}{(1+|z|)^3} \le |f'(z)| \le \frac{1+|z|}{(1-|z|)^3},$$

ii)
$$\frac{|z|}{(1+|z|)^2} \le |f(z)| \le \frac{|z|}{(1-|z|)^2}.$$

*Estimates are sharp and the Koebe function is the only extremal (up to a rotation).*

*Proof.* By rotational invariance, wlog, setting $z = x \in [0,1)$.

$$g(z) = \log f'(z) = \log(1 + 2a_2 z + \cdots) = 2a_2 z + \cdots ,$$

$g(0) = 0$ and $g' = f''/f'$. By Lemma 1.6,

$$\left| \frac{f''(x)}{f'(x)} - \frac{2x}{1-x^2} \right| \le \frac{4}{1-x^2}.$$

By integration,

$$\left| g(x) - \log \frac{1}{1-x^2} \right| \le 2 \log \frac{1+x}{1-x}, \qquad x \in [0,1).$$

So

$$\log \frac{1}{1-x^2} - 2\log\frac{1+x}{1-x} \le \log|f'(x)| \le \log\frac{1}{1-x^2} + 2\log\frac{1+x}{1-x},$$

i.e.

$$\log\frac{1-x}{(1+x)^3} \le \log|f'(x)| \le \log\frac{1+x}{(1-x)^3}.$$

Exponentiating, the first inequality follows.

$$|f(x)| = \left| \int_0^x f'(t)dt \right| \le \int_0^x \frac{1+t}{(1-t)^3} dt = \frac{x}{(1-x)^2}.$$

The upper bound in ii) follows. For the lower bound, set $r \in (0,1)$, $m = \min_{|z|=r} |f(z)| > 0$. Wlog, we can assume $f(re^{i\theta}) = m$ for some $\theta$. Let $\gamma(t) = re^{it}, t \in [0, 2\pi]$. $f \circ \gamma$ does not meet $B(0,m)$. For any $w \in B(0,m)$, by the Argument Principle,

$$\begin{aligned} &\# \text{ of zeros of } f - w \text{ in } B(0,r) \\ &\quad = \mathrm{ind}_{f\circ\gamma}(w) \equiv \mathrm{ind}_{f\circ\gamma}(0) = \# \text{ of zeros of } f - 0 = f \text{ in } B(0,r) = 1. \end{aligned}$$

It follows

Figure 5:

$$B(0,m) \subseteq f(B(0,r)), \quad \text{and} \quad \overline{B}(0,m) \subseteq f(\overline{B}(0,r)) \subseteq \Omega := f(\mathbb{D}),$$

and so $[0,m] \subseteq \Omega$. Let $\alpha(t) = f^{-1}(t), t \in [0,m]$. Then $\alpha(t)$ is a path in $\mathbb{D}$ from $0 = f^{-1}(0)$ to $re^{i\theta} = f^{-1}(m)$.

$$f(\alpha(t)) \equiv t \quad \Longrightarrow \quad f'(\alpha(t))\alpha'(t) \equiv 1.$$

So

$$m = \int_0^m dt = \int_0^m |f'(\alpha(t))||\alpha'(t)|dt = \int_\alpha |f'(z)||dz| = \int_0^L |f'(\tilde{\alpha}(s))|ds,$$

where $\tilde{\alpha} : [0, L] \to \mathbb{C}$ is the arc-length reparametrization of $\alpha$, $L = \ell(\alpha) :=$ length of $\alpha \geq r$, $\tilde{\alpha}(\ell(\alpha([0, t]))) = \alpha(t)$, and

$$\int_\alpha g(z)|dz| = \int_0^L g(\tilde{\alpha}(s))ds.$$

Since $\tilde{\alpha}(0) = \alpha(0) = 0$, $|\tilde{\alpha}(s)| \leq s$. So

$$m = \int_0^L |f'(\tilde{\alpha}(s))|ds \geq \int_0^L \frac{1 - |\tilde{\alpha}(s)|}{(1 + |\tilde{\alpha}(s)|)^3}ds \geq \int_0^r \frac{1 - s}{(1 + s)^3}ds = \frac{r}{(1 + r)^2}. \qquad \square$$

**Corollary 1.8.** *$\mathcal{S}$ is a normal family, i.e. every sequence $\{f_n\}$ in $\mathcal{S}$ has a subsequence $\{f_{n_k}\}$ that converges locally uniformly in $\mathbb{D}$. Moreover, every locally uniform limit of a sequence in $\mathcal{S}$ also lies in $\mathcal{S}$. (So $\mathcal{S}$ is compact with respect to the topology of locally uniform convergence.)*

*Proof.* By Koebe's Distortion Theorem, (up bound in ii)), $\mathcal{S}$ is locally uniform bounded. Hence, $\mathcal{S}$ is a normal family by Montel's Little Theorem. If $\{f_n\}$ is a sequence in $\mathcal{S}$ and $f_n \to f$ locally uniformly on $\mathbb{D}$. Then $f$ is holomorphic (Weierstrass), and constant or injective (Hurwitz). Moreover, $f_n(0) \to f(0)$ and $f_n'(0) \to f'(0)$ which implies $f(0) = 0$ and $f'(0) = 1$. So $f$ is non-constant, hence injective. So $f \in \mathcal{S}$. $\square$

**Remark 1.9.** Koebe's Distortion Theorem often gives useful (non-sharp) quantitative information:

i) Let $\Omega, \Omega' \subsetneq \mathbb{C}$ be two regions, $f : \Omega \to \Omega'$ be conformal map, $z_0 \in \Omega$. Then

$$|f'(z_0)| \simeq \frac{\operatorname{dist}(f(z_0), \partial\Omega')}{\operatorname{dist}(z_0, \partial\Omega)}$$

with universal constant. Where $A \simeq B$ means that there exists a constant $C$ such that

$$\frac{1}{C}A \leq B \leq CA.$$

*Proof.* Let $d' = \operatorname{dist}(f(z_0), \partial\Omega')$, $d = \operatorname{dist}(z_0, \partial\Omega)$. Then $B(z_0, d) \subseteq \Omega$. By 1/4-Theorem (applied to $u \mapsto \frac{f(z_0+ud)-f(z_0)}{f'(z_0)d}$), we have

$$B(f(z_0), \frac{1}{4}|f'(z_0)|d) \subseteq \Omega'.$$

So

$$d' \geq \frac{1}{4}|f'(z_0)|d, \quad \text{and} \quad |f'(z_0)| \leq 4\frac{d'}{d}.$$

For lower bound, consider $f^{-1}$. $\square$

ii) Let $\Omega, \Omega'$ be two regions, $f : \Omega \to \Omega'$ be a conformal map, $K \subseteq \Omega$ be a compact set. Then

$$|f'(z)| \simeq |f'(w)|$$

for any $z, w \in K$ with implicit constant only depending on $\Omega, K$ (and not on $f$!).

*Idea of Proof.* If $\Omega = \mathbb{D}$, then $|f'(z)| \simeq |f'(0)| \simeq |f'(w)|$ by Koebe. Generalize to $\Omega =$ disk. General case follows from Harnack chain argument. $\square$

## 2 Boundary extensions of conformal maps

Suppose $\Omega \subseteq \mathbb{C}$ is a bounded region. Then the following are equivalent (TFAE):
i) $\Omega$ is simply connected;
ii) $\hat{\mathbb{C}} \setminus \Omega$ is connected ($\iff \mathbb{C} \setminus \Omega$ connected);
iii) $\partial\Omega$ is connected;
iv) $\Omega$ is conformally equivalent to $\mathbb{D}$, i.e., there exists a conformal map $f : \mathbb{D} \leftrightarrow \Omega$.

**Theorem 2.1.** *Let $f : \mathbb{D} \to \Omega$ be a conformal map onto a bounded (simply connected) region. TFAE*
i) *$f$ has a continuous extension to $\overline{\mathbb{D}}$;*
ii) *$\partial\Omega$ can be parameterized as a loop, i.e., there exists a continuous map $\varphi : \partial\mathbb{D} \to \mathbb{C}$ such that $\varphi(\partial D) = \partial\Omega$;*
iii) *$\partial\Omega$ is locally connected;*
iv) *$\mathbb{C} \setminus \Omega$ is locally connected.*

We will prove this in the following:

**2.2. Locally connected sets**
Let $A \subseteq \mathbb{C}$ be a closed set. $A$ is locally connected iff for all $a \in A$ and $\varepsilon > 0$, there exists $\delta > 0$ such that if $b \in A$ is arbitrary and $|a - b| < \delta$, then there exists a continuum $E \subseteq A$ with $a, b \in E$ and $\operatorname{diam}(E) < \varepsilon$.

If $A \subseteq \mathbb{C}$ is a compact set, then $A$ is locally connected iff for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $a, b \in A$ with $|a - b| < \delta$, then there exists a continuum $E \subseteq A$ with $a, b \in E$ and $\operatorname{diam}(E) < \varepsilon$.

*Proof.* $\Longleftarrow$ trivial.
$\Longrightarrow$ By contradiction. If not, there exist $\varepsilon_0 > 0$ ("bad $\varepsilon$") and sequences $\{a_n\}, \{b_n\}$ in $A$ such that $|a_n - b_n| \to 0$ but no continuum $E$ such that $a_n, b_n \in E$ and $\operatorname{diam} E < \varepsilon_0$. Wlog, assume $a_n, b_n \to c$.
Since $A$ is locally connected, for sufficiently large $n$, there exist continuums $E_n'$, $E_n''$ such that $a_n, c \in E_n'$, $b_n, c \in E_n''$, $\operatorname{diam}(E_n') < \varepsilon_0/2$, $\operatorname{diam}(E_n'') < \varepsilon_0/2$. Then $E_n = E_n' \cup E_n''$ is a continuum with $a_n, b_n \in E_n$ and $\operatorname{diam}(E_n) < 2 \cdot \varepsilon_0/2 = \varepsilon_0$, a contradiction! □

A compact set $A \subseteq \mathbb{C}$ is locally connected, iff points that are close have a small connection, iff there exists $\omega : (0, \infty) \to (0, \infty)$ with $\lim_{\delta \to 0^+} \omega(\delta) = 0$ such that $\forall a, b \in A$, $\exists$ continuum $E \subseteq A$ with $a, b \in E$ and $\operatorname{diam}(E) \leq \omega(|a - b|)$.
Boundary of comb domain is connected but not locally connected.

Figure 6: Comb domain

Let $A \subseteq \mathbb{C}$ be compact and locally connected, $\varphi : A \to \mathbb{C}$ continuous, and $B := \varphi(A)$. Then $B$ is locally connected. (Continuous images of compact and locally connected sets are locally connected.)

*Proof.* By contradiction! If not, then there exist $\varepsilon_0 > 0$ and sequences $\{b_n\}, \{b_n'\}$ such that $|b_n - b_n'| \to 0$ but there exist no continuum $E \subseteq B$ with $b_n, b_n' \in E$, $\operatorname{diam}(E) < \varepsilon_0$. There exist

$a_n, a'_n$ such that $b_n = \varphi(a_n)$, $b'_n = \varphi(a'_n)$. Wlog, $a_n \to x$ and $a'_n \to y$. Then $b_n, b'_n \to z = \varphi(x) = \varphi(y)$. We can find small connections $E'_n$ and $E''_n$ between $x, a_n$ and $y, a'_n$ (resp.) for $n$ large. Then $F_n = \varphi(E'_n) \cup \varphi(E''_n)$ is a small connection between $b_n$, $b'_n$ for $n$ large, by uniform continuity of $\varphi$. Contradiction! $\square$

In particular, if $\varphi : \partial\mathbb{D} \to \mathbb{C}$ is conformal, then $\varphi(\partial\mathbb{D})$ is locally connected. (Loops or pathes are locally connected.) So ii) $\implies$ iii) in Theorem 2.1!

**Lemma 2.3. (Wolff's Lemma)** *Let $U \subseteq \mathbb{C}$ be open, $f : U \to V \subseteq B(0, R_0)$ be conformal, $z_0 \in \overline{U}$, $C(r) := U \cap \{z \in \mathbb{C} : |z - z_0| = r\}$. Then*

$$\inf_{\rho < r < \sqrt{\rho}} \ell(f(C(r))) \leq \frac{2\pi R_0}{\sqrt{\log(1/\rho)}}, \quad \text{for} \quad 0 < \rho < 1.$$

*In particular, there exists a sequence $r_n \to 0$ such that*

$$\ell(f(C(r_n))) \to 0 \qquad \text{as} \quad n \to \infty.$$

(If a "thick" family of curves is confined to a set of controlled area, then one of the curves has to be short.)

Figure 7:

*Proof.* Let $L(r) := \ell(f(C(r)))$ (lower semi-continuous). Then

$$\begin{aligned} L(r)^2 &= \left( \int_{C(r)} |f'(z)||dz| \right)^2 \\ &\leq \left( \int_{C(r)} |dz| \right) \left( \int_{C(r)} |f'(z)|^2 |dz| \right) \qquad \text{(Schwarz inequality)} \\ &\leq 2\pi r \int_{\{t \in [0,2\pi] : z_0 + re^{it} \in U\}} |f'(z_0 + re^{it})|^2 r\, dt \end{aligned}$$

So

$$\int_0^\infty \frac{L(r)^2}{r} dr \leq 2\pi \int_U |f'(z)|^2 dA(z) = 2\pi \operatorname{Area}(V) \leq 2\pi^2 R_0^2.$$

This gives

$$\frac{1}{2} \log \frac{1}{\rho} \inf_{\rho < r < \sqrt{\rho}} L(r)^2 \leq \int_\rho^{\sqrt{\rho}} L(r)^2 \frac{dr}{r} \leq 2\pi^2 R_0^2.$$

The claim follows. $\square$

**Lemma 2.4.** *Let $\gamma : [0, 1) \to \mathbb{C}$ be a path with the length*

$$\ell(\gamma) = \sup_{0 \leq t_0 < \cdots < t_n < 1} \sum_{k=1}^n |\gamma(t_k) - \gamma(t_{k-1})| < \infty.$$

*Then $\lim_{t \to 1^-} \gamma(t)$ exists.*

(If a path has finite length, then it ends some where!)

*Proof.* Denote $L := \ell(\gamma) < \infty$, $L(t) := \ell(\gamma|[0,t])$. Then $L(t) \nearrow L$ as $t \to 1^-$, and so $\ell(\gamma|(t,1)) = L - L(t) \to 0$ as $t \to 1^-$. So for $s, s' \in (t,1)$

$$|\gamma(s) - \gamma(s')| \leq \ell(\gamma|(t,1)) \to 0 \quad \text{as} \quad t \to 1^-.$$

This implies that for every sequence $\{s_n\}$ in $[0,1)$ with $s_n \to 1$, $\{\gamma(s_n)\}$ is a Cauchy sequence. The claim follows. □

Let $A \subseteq \mathbb{C}$ be a closed set, and $x, y \in \mathbb{C}$. We say that $A$ *separates* $x$ and $y$ if $x, y$ do not lie in one component of $\mathbb{C} \setminus A$ (true if $x \in A$ or $y \in A$!). It is equivalent to that every path joining $x, y$ meets $A$.

**Janiszewski's Theorem.** *Suppose that $K, L \subseteq \mathbb{C}$ are compact sets such that $K \cap L$ connected. If $K \cup L$ separates two points $x, y \in \mathbb{C}$, then they are separated by $K$ or by $L$.*

**Lemma 2.5.** *Let $K \in \mathbb{D}$ be compact, $x_0 \in \mathbb{C}$ such that* $\operatorname{dist}(x_0, K) > \operatorname{diam}(K)$*, $u, v \in \mathbb{C}$. If $K$ separates $x_0$ and $u$, and separates $x_0$ and $v$, them* $|u - v| \leq \operatorname{diam}(K)$.

Figure 8: Proof of the lemma, $u \neq v$.

*Proof.* Pick $a \in K$ and let $R = \operatorname{diam}(K)$. Then $K \subseteq \overline{B}(a, R)$ and $|x_0 - a| > R$. So $x_0 \in \mathbb{C} \setminus \overline{B}(a, R) \subseteq \mathbb{C} \setminus K$. This shows that $x_0$ lies in the unbounded component of $\mathbb{C} \setminus K$.

So both $u, v$ do not lie in the unbounded component of $\mathbb{C} \setminus K$. This implies if $\ell \in \mathbb{C}$ is the line with $u, v \in \ell$, then there exist $u', v' \in K$ such that $[u, v] \subseteq [u', v']$. Hence, $|u - v| \leq |u' - v'| \leq \operatorname{diam}(K)$. □

*Proof of Theorem 2.1.* i) $\Longrightarrow$ ii).

Suppose $f$ has a continuous extension $f : \overline{\mathbb{D}} \to \mathbb{C}$. By continuity, $f(\overline{\mathbb{D}}) \subseteq \overline{f(\mathbb{D})} = \overline{\Omega}$. By compactness of $\mathbb{D}$, $\overline{\Omega} = \overline{f(\mathbb{D})} \subseteq f(\overline{\mathbb{D}})$. So $\overline{\Omega} = f(\overline{\mathbb{D}})$. Since $\Omega = f(\mathbb{D})$ is open, $\partial\Omega = \overline{\Omega} \setminus \Omega \subseteq f(\partial\mathbb{D})$. Moreover, conformality implies $f(\partial\mathbb{D}) \subset \overline{\Omega} \setminus \Omega = \partial\Omega$. So $f(\partial\mathbb{D}) = \partial\Omega$, which implies that $\partial\Omega$ has a parametrization as a loop.

ii) $\Longrightarrow$ iii).

Continuous images of compact, locally connected sets are locally connected (see 2.2). Since $\partial\mathbb{D}$ is compact and locally connected, $\partial\Omega = f(\partial\mathbb{D})$ also has these properties.

iii) $\Longrightarrow$ iv).

Let $u, v \in \mathbb{C} \setminus \Omega$ be two arbitrary points. Run along $[u, v]$:

1) If $[u, v] \cap \partial\Omega = \varnothing$, then $[u, v]$ is a continuum in $\mathbb{C} \setminus \Omega$ joining $u, v$ with $\operatorname{diam}(E) = |u - v|$.

Figure 9:

By assumption, there exists a continuum $E' \subseteq \partial\Omega$ with $u', v' \in E'$ and 2) If $[u, v] \cap \partial\Omega \neq \varnothing$, then we can find $u', v' \in \partial\Omega$ such that $[u, u'] \subseteq \mathbb{C} \setminus \Omega$, $[v', v] \subseteq \mathbb{C} \setminus \Omega$. $\operatorname{diam}(E') \leq \omega(|u' - v'|)$ where $\omega(\delta) \to 0$ as $\delta \to 0^+$. Then $E := [u, u'] \cup E' \cup [v', v]$ is a continuum with $E \subseteq \mathbb{C} \setminus \Omega$, $u, v \in E$, and

$$\operatorname{diam}(E) \leq |u - v| + \omega(|u' - v'|) \leq |u - v| + \omega(|u - v|) = \tilde{\omega}(\delta),$$

where $\tilde{\omega}(\delta) = \delta + \omega(\delta)$ and $\delta = |u - v|$. Since $\tilde{\omega}(\delta) \to 0$ as $\delta \to 0^+$, the claim follows.

iv) $\Longrightarrow$ i).

It is sufficient to show that $f$ is uniformly continuous on $\mathbb{D}$, i.e., there exists an $\omega : (0, \infty) \to (0, \infty)$ with $\omega(\delta) \to 0$ as $\delta \to 0^+$ such that

$$|f(x) - f(y)| \leq \omega(|x - y|), \qquad \text{for all} \quad x, y \in \mathbb{D}.$$

(then the image of every Cauchy sequence is Cauchy, bla, bla, bla, ...) equivalently,

$$\operatorname{diam}(f(B(z_0, \delta) \cap \mathbb{D})) \leq \omega(\delta), \qquad \text{for} \quad z_0 \in D, \ \delta > 0.$$

Here, wlog, $\delta > 0$ is small and $z_0 \in \mathbb{D}$ is close to $\partial\mathbb{D}$. By translation and scaling of $\Omega$, wlog, we can assume $f(0) = 0$, $z_0 \in \mathbb{D}$, $w_0 = f(z_0)$ satisfying $|z_0|, |w_0| \geq 1/2$.

Figure 10:

By Wolff's Lemma 2.3, there exists $r \in (\delta, \sqrt{\delta})$ such that

$$\ell(f(C)) \leq \omega_1(\delta),$$

where $C = \mathbb{C} \cap \{z \in \mathbb{C} : |z - z_0| = r\}$, $\omega_1 = C_0/\sqrt{\log(1/\delta)} \to 0$ as $\delta \to 0$ (for some constant $C_0 > 0$).

Let us assume $C$ is not the whole circle $|z - z_0| = r$, but an open subarc. Then Lemma 2.4 implies that $f(C)$ has two end points $u, v \in \partial\Omega$. So $A := \overline{f(C)} = f(C) \cup \{u, v\}$ (possibly $u = v$). Then $|u - v| \leq \ell(f(C)) \leq \omega_1(\delta)$. Since $\mathbb{C} \setminus \Omega \supset \partial\Omega$ is locally connected, there exists a continuum $B \subseteq \mathbb{C} \setminus \Omega$ such that $u, v \in B$ and

$$\operatorname{diam}(B) \leq \omega_2(|u - v|) \leq \omega_3(\delta).$$

Let $K = A \cup B$. Then

$$\operatorname{diam}(K) \leq \operatorname{diam}(A) + \operatorname{diam}(B) \leq \omega_1(\delta) + \omega_3(\delta) = \omega_4(\delta),$$

and $K \cap \partial\Omega \neq \varnothing$. So $\operatorname{dist}(a, K) > \operatorname{diam}(K)$ if $\delta$ is small enough.

Figure 11:

Now let $z \in B(z_0, \delta) \cap \mathbb{D}$ be arbitrary and $w = f(z)$. Then $C$ separates 0 and $z$ in $\mathbb{D}$, i.e., $(\mathbb{C} \setminus \mathbb{D}) \cup C$ separates 0 and $z$. This implies $(\mathbb{C} \setminus \Omega) \cup (f(C) \cup B)$ separates $0 = f(0)$ and $w = f(z)$. Since $(\mathbb{C} \setminus \Omega) \cap (f(C) \cup B) = B$ is connected, and $\mathbb{C} \setminus \Omega$ does not separate 0 and $w$, we get $K = f(C) \cup B$ separates 0 and $w$ by Janiszewski's Theorem. If $z' \in B(z_0, \delta) \cap \mathbb{D}$ is another point and $w' = f(z')$, then $K$ separates 0 and $w'$ by the same argument. Lemma 2.5 implies

$$|w - w'| \leq \operatorname{diam}(K) \leq \omega_4(\delta),$$

and so

$$\operatorname{diam}(f(B(z_0, \delta) \cap \mathbb{D})) \leq \omega_4(\delta),$$

as desired. □

**Remark 2.6.** A similar argument shows that if $f : \mathbb{D} \to \Omega \subseteq \hat{\mathbb{C}}$ is conformal, then $f$ has a continuous extension $f : \overline{\mathbb{D}} \to \overline{\Omega} \subseteq \hat{\mathbb{C}}$ if $\partial\Omega$ (or $\hat{\mathbb{C}} \setminus \Omega$) is locally connected. Here, we use spherical or chordal distance in the target! (Versions of Wolff's Lemma and Lemma 2.5 still true for spherical metric.)

Let $K$ be a continuum. A point $p$ is a cut point of $K$ if $K \setminus \{p\}$ is not connected.

**Proposition 2.7.** *Let $\Omega \subseteq \mathbb{D}$ be a bounded simply connected region, $f : \mathbb{D} \to \Omega$ be a conformal map with continuous extension $f : \overline{\mathbb{D}} \to \overline{\Omega}$. Let $p \in \partial\Omega$. Then $\# f^{-1}(p) \geq 2$ if and only if $p$ is a cut point of $\partial\Omega$.*

*More precisely, let $A := f^{-1}(p) \subseteq \partial\mathbb{D}$, and $\partial\mathbb{D} \setminus A = \bigcup_{k\in\Lambda} I_k$ be the decomposition into pairwise disjoint open arcs ($\Lambda$ countable indexes set). Then the sets $f(I_k)$, $k \in \Lambda$, form the pairwise disjoint connected components of $\partial\Omega \setminus \{p\}$. (Note that $\#\Lambda = \#A$, so $\#\Lambda \geq 2$ iff $\#A \geq 2$.)*

*Proof.* Note that $\partial\Omega \setminus \{p\} = f(\partial\mathbb{D} \setminus A) = \bigcup_{k\in\Lambda} f(I_k)$, and the sets $f(I_k)$ are connected (conformal images of connected sets!). It suffices to show that $f(I_k)$, $k \in \Lambda$, are pairwise disjoint. Let $I, I'$ be two of these arcs, and $C$ the circular arc in $\mathbb{D}$ with the same end points as $I$. Then $C$ divides $\mathbb{D}$ into two parts $D$ and $D'$ such that $I \subseteq \partial D$, $I' \subseteq \partial D'$ and $\mathbb{D} = D \cup C \cup D'$ is a disjoint union.

Figure 12:

Let $J = f(C) \cup \{p\}$, $U = f(D)$ and $U' = f(D')$. Then $J$ is a Jordan curve, and $U, U'$ are open connected set in $\mathbb{C} \setminus J$. So $U \subseteq \mathrm{In}(J)$ or $U \subseteq \mathrm{Out}(J)$; and $U' \subseteq \mathrm{In}(J)$ or $U' \subseteq \mathrm{Out}(J)$. We say $U, U'$ can not lie in the same component of $\mathbb{C} \setminus J$.

Suppose $U, U' \subseteq \mathrm{In}(J)$. By the open mapping theorem, $U \cup f(C) \cup U' = \Omega$ is an open neighborhood of each point on $f(C) \subseteq J$. On the other hand, $\mathrm{Out}(J)$ is disjoint from $U \cup f(C) \cup U'$ by the assumption. But $\partial\,\mathrm{Out}(J) = J$ which implies that $\mathrm{Out}(J)$ contains points near $J$. A contradiction.

So $U, U'$ lie in different components of $\mathbb{C} \setminus J$, say, $U \subseteq \mathrm{In}(J)$, $U' \subseteq \mathrm{Out}(J)$. Then, $f(I) \subseteq f(\overline{D}) \subseteq \overline{U} \subseteq J \cup \mathrm{In}(J)$. On the other hand, $f(I) \subseteq \partial\Omega \setminus \{p\}$, and $\partial\Omega \setminus \{p\} \cap J = \varnothing$. So $f(I) \subseteq \mathrm{In}(J)$. Similarly, $f(I') \subseteq \mathrm{Out}(J)$. Hence, $f(I) \cap f(I') \subseteq \mathrm{In}(J) \cap \mathrm{Out}(J) = \varnothing$. □

**Theorem 2.8. (Carathéodory)** *Let $f : \mathbb{D} \to \Omega$ be a conformal map onto a bounded simply connected region. TFAE*

i) *$f$ has a homeomorphic extension to $\overline{\mathbb{D}}$ (i.e., continuous and injective).*

ii) *$\partial\Omega$ is a Jordan curve.*

iii) *$\partial\Omega$ is locally connected and has no cut points.*

*Proof.* i) $\Longrightarrow$ ii) Obvious, because $\partial\Omega = f(\partial\mathbb{D})$.

ii) $\Longrightarrow$ iii) Clear.

iii) $\Longrightarrow$ i)

By Theorem 2.1, $f$ has a continuous extension $f : \overline{\mathbb{C}} \to \overline{\Omega}$. By Proposition 2.7, $f|\partial\mathbb{D}$ is injective. Since $\partial\Omega = f(\partial\mathbb{D})$ and $\Omega = f(\mathbb{D})$ are disjoint, $f$ is injective on $\overline{\mathbb{D}}$. □

A region $\Omega \subseteq \hat{\mathbb{C}}$ is called an (open) Jordan region or domain if $\partial\Omega \subseteq \hat{\mathbb{C}}$ is a Jordan curve. If $\partial\Omega \subseteq \mathbb{C}$ (i.e., $\infty \notin \partial\Omega$), then $\Omega = \mathrm{In}(\partial\Omega)$ or $\Omega = \mathrm{Out}(\partial\Omega) \cup \{\infty\}$. A closed Jordan region is the closure $\overline{\Omega}$ of an open Jordan region $\Omega \subseteq \hat{\mathbb{C}}$. An open Jordan region is simply connected, because $\partial\Omega$ is connected.

**Corollary 2.9.** *Let $\Omega, \Omega' \subseteq \hat{\mathbb{C}}$ be Jordan regions, $f : \Omega \leftrightarrow \Omega'$ be a conformal map. Then $f$ has a (unique) homeomorphic extension $f : \overline{\Omega} \leftrightarrow \overline{\Omega'}$ (w.r.t. chordal metric on $\hat{\mathbb{C}}$).*

*Proof.* Wlog, $\Omega, \Omega' \subseteq \mathbb{C}$ (use Möbius transform). There exists a conformal map $g : \mathbb{D} \to \Omega$. Then $h := f \circ g : \mathbb{D} \to \Omega'$ is a conformal map. By Theorem 2.8, $g$ and $h$ have homeomorphic extensions $\overline{g} : \overline{\mathbb{D}} \leftrightarrow \overline{\Omega}$, $\overline{h} : \overline{\mathbb{D}} \leftrightarrow \overline{\Omega'}$ respectively. Then $\overline{f} := \overline{h} \circ \overline{g}^{-1} : \overline{\Omega} \leftrightarrow \overline{\Omega'}$ is a homeomorphic extension of $f$. □

**Lemma 2.10.** *Let $\varphi : \partial\mathbb{D} \to \partial\mathbb{D}$ be a homeomorphism. Then $\varphi$ can be extended to a homeomorphism $\overline{\varphi} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.*

*Proof.* Use "radial" extension. Let $\overline{\varphi}(r \cdot \xi) = r \cdot \varphi(\xi)$, where $0 \leq r < \infty$, $\xi \in \partial\mathbb{D}$, and $\overline{\varphi}(\infty) = \infty$. This is a continuous bijection with continuous inverse (= radial extension of $\varphi^{-1}$). Furthermore, $\overline{\varphi}|\overline{\mathbb{D}} : \overline{\mathbb{D}} \leftrightarrow \overline{\mathbb{D}}$ is a homeomorphic extension of $\varphi$. □

**Theorem 2.11.** *Let $f : \mathbb{D} \to \Omega$ be a conformal map onto a Jordan region $\Omega \subseteq \hat{\mathbb{C}}$. Then $f$ has a homeomorphic extension $\overline{f} : \hat{\mathbb{C}} \leftrightarrow \hat{\mathbb{C}}$.*

*Proof.* Wlog, assume $J := \partial\Omega \subseteq \mathbb{C}$, $\Omega = \mathrm{In}(J)$. Then $f$ has a homeomorphic extension $f : \overline{\mathbb{D}} \leftrightarrow \overline{\Omega}$. Note that $\tilde{\mathbb{D}} = \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ and $\tilde{\Omega} = \hat{\mathbb{C}} \setminus \overline{\Omega}$ are two Jordan regions. So there exists a conformal map $\tilde{f} : \tilde{\mathbb{D}} \to \tilde{\Omega}$ with homeomorphic extension $\tilde{f} : \overline{\tilde{\mathbb{D}}} \leftrightarrow \overline{\tilde{\Omega}}$. If $f|\partial\mathbb{D} = \tilde{f}|\partial\mathbb{D}$, then $f, \tilde{f}$ would post together to homeomorphic extension of $f$. However, it is not true in general!

Let $\varphi := \tilde{f}^{-1} \circ f|\partial\mathbb{D}$ ("conformal welding map induced by $J$"). Then $\varphi$ is a homeomorphism on $\partial\mathbb{D}$. By Lemma 2.10, it has a homeomorphic extension $\overline{\varphi} : \hat{\mathbb{C}} \leftrightarrow \hat{\mathbb{C}}$. Define

$$\overline{f} = \begin{cases} f(z) & \text{for} \quad z \in \overline{\mathbb{D}}, \\ \tilde{f}(\overline{\varphi}(z)) & \phantom{\text{for}} \quad z \in \hat{\mathbb{C}} \setminus \mathbb{D}. \end{cases}$$

This is well-defined, and a homeomorphism $\hat{\mathbb{C}} \leftrightarrow \hat{\mathbb{C}}$, which extends $f$. □

**Theorem 2.12. (Schönflies)** *Every homeomorphism $\varphi : J \leftrightarrow J'$ between Jordan curves can be extended to a homeomorphism $\overline{\varphi} : \hat{\mathbb{C}} \leftrightarrow \hat{\mathbb{C}}$. In particular, every Jordan curve $J \subseteq \hat{\mathbb{C}}$ is the image of $\partial\mathbb{D}$ under a homeomorphism $\overline{\varphi} : \hat{\mathbb{C}} \leftrightarrow \hat{\mathbb{C}}$.*

*Proof.* Wlog, assume $J = \partial\mathbb{D}$ and $J' \subseteq \mathbb{C}$. Let $\Omega = \mathrm{In}(J')$. There exists a conformal map $f : \mathbb{D} \leftrightarrow \Omega$ with homeomorphic extension $\overline{f} : \hat{\mathbb{C}} \leftrightarrow \hat{\mathbb{C}}$ (Theorem 2.11). Let $\psi = (f|\partial\mathbb{D})^{-1} \circ \varphi$. This is a homeomorphism $\psi : \partial\mathbb{D} \leftrightarrow \partial\mathbb{D}$, and so has a homeomorphic extension $\overline{\psi} : \hat{\mathbb{C}} \leftrightarrow \hat{\mathbb{C}}$. Then $f \circ \overline{\psi}$ is a homeomorphism $\hat{\mathbb{C}} \leftrightarrow \hat{\mathbb{C}}$ with $f \circ \overline{\psi}|\partial\mathbb{D} = f \circ \psi = f \circ (f|\partial\mathbb{D})^{-1} \circ \varphi = \varphi$. □

### 2.13. Orientation

Let $z_1, z_2, z_3 \in \partial\mathbb{D}$ be three distinct points. This triple is in *positive cyclic order* if in the standard parametrization $\gamma : \mathbb{R} \to \partial\mathbb{D}$, $\gamma(t) = e^{it}$, whenever $\gamma(t_1) = z_1$ and $t_2, t_3 \in (t_1, t_1 + 2\pi)$ with $\gamma(t_2) = z_2$, $\gamma(t_3) = z_3$, we have $t_2 < t_3$.

Note that every $\varphi \in \mathrm{Aut}(\mathbb{D})$ preserves the positive cyclic order of points on $\partial\mathbb{D}$.

The triple $z_1, z_2, z_3 \in \partial\mathbb{D}$ is positive oriented iff $\mathrm{Im}(u, z_1, z_2, z_3) < 0$ for $u \in \mathbb{D}$ ($\mathbb{D}$ lies to the left of $\partial\mathbb{D}$).

**Positive cyclic order on boundary of Jordan region:**

Let $\Omega \subseteq \hat{\mathbb{C}}$ be a Jordan region, $w_1, w_2, w_3 \in \partial\Omega$ are distinct points. $w_1, w_2, w_3$ are in *positive cyclic order* if the following is true: If $f$ is a conformal map $f : \mathbb{D} \leftrightarrow \Omega$ with homeomorphic extension $f : \overline{\mathbb{D}} \leftrightarrow \overline{\Omega}$. Let $z_k = f^{-1}(w_k)$, $k = 1, 2, 3$. The requirement is that $z_1, z_2, z_3$ are in positive cyclic order on $\partial\mathbb{D}$.

The definition is independent of the choice of $f$. Let $g : \mathbb{D} \leftrightarrow \Omega$ be another conformal map with homeomorphic extension $g : \overline{\mathbb{D}} \leftrightarrow \overline{\Omega}$. Let $z'_k = g^{-1}(w_k)$, $k = 1, 2, 3$. Since $\varphi = f^{-1} \circ g \in \mathrm{Aut}(\mathbb{D})$, we get $z_1, z_2, z_3$ in positive cyclic order iff $z'_1, z'_2, z'_3$ in positive cyclic order.

**Theorem 2.14.** *Let $\Omega, \Omega' \subseteq \hat{\mathbb{C}}$ be two Jordan regions, $z_1, z_2, z_3$ in positive cyclic order on $\partial\Omega$, $w_1, w_2, w_3$ in positive cyclic order on $\partial\Omega'$. Then there exists a unique conformal map $f : \Omega \leftrightarrow \Omega'$ whose homeomorphic extension $f : \overline{\Omega} \leftrightarrow \overline{\Omega'}$ satisfies $w_k = f(z_k)$, $k = 1, 2, 3$.*

*Proof.* Pull back by auxiliary conformal maps, we can assume that $\Omega = \mathbb{D}$, $\Omega' = \mathbb{D}$ (see figure) Then the existence and the uniqueness follow from the fact that there exists a unique Möbius transform $\varphi \in \mathrm{Aut}(\mathbb{D})$ with $w'_k = \varphi(z'_k)$. □

Figure 13: pull back

**Example 2.15.** Let $f : \mathbb{D} \to \Omega$ be a conformal map onto the "slit disk" $\Omega = \mathbb{D} \setminus [0, 1)$. $\partial\Omega$ is locally connected. So there exists continuous extension $f : \overline{\mathbb{D}} \to \overline{\Omega}$. Since $\partial\Omega \setminus \{1\}$ has two components, so by Proposition 2.7, $\# f^{-1}(1) = 2$. Let $f^{-1} = \{a, b\}$. $\partial\mathbb{D} \setminus \{a, b\} = I_1 \cup I_2$ such that $f(I_1) = \partial\mathbb{D} \setminus \{1\}$ and $f(I_2) = [0, 1)$. Since $\partial\mathbb{D} \setminus \{1\}$ has not cut points, $\# f^{-1}(p) = 1$ for $p \in \partial\mathbb{D} \setminus \{1\}$. So $f : I_1 \to \partial\mathbb{D} \setminus \{1\}$ is a homeomorphism. Since $\# f^{-1}(0) = 1$, so there exists unique $c \in I_2$ such that $f(c) = 0$.

Figure 14: example

**Lemma 2.16.** *Let $\Omega \subseteq \mathbb{C}$ be a simply connected region, $z_0 \in \Omega$ be a base point, $D \subseteq \mathbb{C}$ be a disk with $C = \partial D$ such that $z_0 \notin D$. $C \cap \Omega = \bigcup_{k \in \{1,2,3,\ldots\}} C_k$, the pairwise disjoint union of circle arcs. If $z \in \Omega \cap D$, then one of the arcs $C_k$ separates $z_0$ and $z$ in $\Omega$ (i.e., every path in $\Omega$ joining $z_0$ and $z$ meets $C_k$).*

*Proof.* Suppose it is not. Then none of compact sets $A_k := \hat{\mathbb{C}} \setminus \Omega \cup C_k$, $k = 1, 2, \ldots$, separates $z_0$ and $z$. There exists a path $\gamma$ in $\Omega$ joining $z_0$ and $z$. It has positive distance to $\partial\Omega$, so it can only meet finitely many arcs $C_k$ ($\overline{C}_k \cap \partial\Omega \neq \varnothing$, and $\operatorname{diam} C_k \to 0$ as $k \to \infty$ if there are infinitely many). So there exists $N \in \mathbb{N}$ such that $B := A_N \cup A_{N+1} \cup \cdots$ does not meet $\gamma$, so $B$ does not separate $z_0$ and $z$. Since $A_1 \cap B = \hat{\mathbb{C}} \setminus \Omega$ is connected, and neither $A_1$ nor $B$ separate $z_0$ and $z$, $A_1 \cup B$ does not separate $z_0$ and $z$ either by Janikovski! Repeating this argument, we see that $A_1 \cup A_2 \cup B$ does not separate $z_0$ and $z$, etc.. So $A_1 \cup \cdots \cup A_{N-1} \cup B = \bigcup_{k \in \{1,2,\ldots\}} A_k \cup \hat{\mathbb{C}} \setminus \Omega = C \cup \hat{\mathbb{C}} \setminus \Omega$ does not separate $z_0$ and $z$. But $C$ separates $z_0, z$. Contradiction! □

Figure 15:

**Theorem 2.17. (Fundamental distortion estimate for conformal maps into $\mathbb{D}$)** *There exists a function (universal distortion function) $\omega : (0, \infty) \to (0, \infty)$, $\omega(\delta) \to 0$ as $\delta \to 0^+$ with the following property: Let $\Omega \subseteq \mathbb{C}$ be a simply connected region, $g : \Omega \to \mathbb{D}$ a conformal map, and $K \subset \Omega$ be a continuum. Then*

$$\operatorname{diam}(g(K)) \le \omega\Big(\frac{\operatorname{diam}(K)}{|f'(0)|}\Big), \tag{5}$$

*where $f = g^{-1} : \mathbb{D} \to \Omega$. One can take $\omega(\delta) = c_0/\sqrt{\log(1/\delta)}$.*

*Proof.* Without lose of generality, we assume $g(0) = 0 = f(0)$, $g'(0) = 1 = f'(0)$. The proof is similar to the proof of Theorem 2.1 using Wolff's Lemma applied to $g'$. Wlog, assume $\operatorname{diam}(K)$ very small.

Note that $f(\overline{B}(0, 1/2)) \supseteq \overline{B}(0, 2/9)$ (follows from lower bounded in Theorem 1.7 and its proof). So $g(\overline{B}(0, 2/9) \subseteq \overline{B}(0, 1/2)$. By Koebe's Distortion Theorem, it follows that $|g'| \le c_0$ on $\overline{B}(0, 2/9)$ with $c_0$ independent of $g$. So $g$ is uniformly Lipschitz on $\overline{B}(0, 2/9)$. (5) follows if $K$ close to 0. Pick $z_0 \in K$. Let $\delta := \operatorname{diam}(K)$. Then $K \subseteq \overline{B}(z_0, \delta)$. By Wolff's Lemma, there exists $r \in (\delta, \sqrt{\delta})$ such that for $C_0 = \{|z - z_0| = r\}$ we have

$$\ell(g(C_0 \cap \Omega)) \le \omega(\delta).$$

We may assume that 0 lies outside $C_0$. By Lemma 2.16, there exists a circular arc $C \subseteq C_0 \cap \Omega$ such that $C$ separates 0 and $z_0$ in $\Omega$. Then $C$ actually separates 0 and every point on $K$ in $\Omega$ since $K$ is connected. Then

$$\ell(g(C)) \le \ell(g(C_0 \cap \Omega)) \le \omega(\delta) \ll 1,$$

and $g(C)$ separates 0 and $g(K)$ in $\mathbb{D}$. Hence

$$\operatorname{diam} g(K) \le 2 \operatorname{diam} g(C) \le 2\omega(\delta).$$

(Note: if $d = \operatorname{diam}(K)$, $w_0 \in g(K)$, and $d$ is small, then $g(K) \subseteq \overline{B}(w_0, d)$.) □

**Definition 2.18.** Let $\Omega \subseteq \mathbb{C}$ be a region, $a, b \in \Omega$. We define

$$\lambda_\Omega(a, b) = \inf_\gamma \ell(\gamma),$$

where inf is taken over all pathes in $\Omega$ joining $a, b$, and

$$\rho_\Omega(a, b) = \inf_K \operatorname{diam}(K),$$

where inf is taken over all continuum $K \subseteq \Omega$ with $a, b \in K$. Both $\lambda_\Omega$ and $\rho_\Omega$ are metrics on $\Omega$, called the *inner length metric* on $\Omega$ and the *diameter metric* on $\Omega$, resp.

Note that $\rho_\Omega \le \lambda_\Omega$, and $\rho_\Omega, \lambda_\Omega$ induce the Euclidean topology on $\Omega$. If $a \in \Omega$ and $b$ is close to $a$, then $\rho_\Omega(a, b) = \lambda_\Omega(a, b) = |a - b|$. If $\Omega$ is a convex region, both $\rho_\Omega$ and $\lambda_\Omega$ agree with the Euclidean metric.

**Corollary 2.19.** *Let $\Omega \subseteq \mathbb{C}$ be a simply connected region and $g : \Omega \to \mathbb{D}$ be a conformal map. Then $g : (\Omega, \rho_\Omega) \to \mathbb{D}$ and $g : (\Omega, \lambda_\Omega) \to \mathbb{D}$ are uniformly continuous, where $\mathbb{D}$ equipped with Euclidean metric.*

*Proof.* Let $w_1, w_2 \in \Omega$ be arbitrary, $K \subseteq \Omega$ be compact with $w_1, w_2 \in K$ with $\operatorname{diam}(K)$ close to $\rho_\Omega(w_1, w_2)$. Let $z_1 = g(w_1)$, $z_2 = g(w_2)$. By Theorem 2.17,

$$|z_2 - z_1| \leq \operatorname{diam} g(K) \leq \tilde{\omega}(\operatorname{diam}(K)) \to \tilde{\omega}(\rho_\Omega(w_1, w_2))$$

as $\operatorname{diam}(K) \to \rho_\Omega(w_1, w_2)$. So

$$|z_2 - z_1| \leq \tilde{\omega}(\rho_\Omega(w_1, w_2)) \leq \tilde{\omega}(\lambda_\Omega(w_1, w_2))$$

(if $\tilde{\omega}$ is increasing as we may assume). □

**Corollary 2.20.** *Let $\Omega \subseteq \mathbb{C}$ be a simply connected region and $g : \Omega \to \mathbb{D}$ be a conformal map. Suppose $\gamma : [0, 1) \to \Omega$ is a path with $\lim_{t\to 1^-} \gamma(t) = w_0 \in \partial\Omega$. Then $\lim_{t\to 1^-} g(\gamma(t)) = z_0 \in \partial\mathbb{D}$ exists.*

*Proof.* Our hypothesis implies that $\operatorname{diam} \gamma([t, 1)) \to 0$ as $t \to 1^-$. By Theorem 2.17, $\operatorname{diam} g \circ \gamma([t, 1)) \to 0$ as $t \to 1^-$. Hence, $\lim_{t\to 1^-} g \circ \gamma(t) = z_0 \in \overline{\mathbb{D}}$ exists. Then $z_0 \in \partial\mathbb{D}$, because otherwise $z_0 \in \mathbb{D}$, and $\gamma(t) = g^{-1}(g(\gamma(t))) \to g^{-1}(z_0) = w_0 \in \Omega$. Contradiction! □

**Remark 2.21.** For every simply connected region $\Omega \subseteq \hat{\mathbb{C}}$, one can introduce a suitable compactification $\hat{\Omega}$ (*prime end compactification*) such that every conformal map $f : \Omega_1 \leftrightarrow \Omega_2$ between simply connected regions extends to a homeomorphism $\hat{f} : \hat{\Omega}_1 \leftrightarrow \hat{\Omega}_2$. (Carathéodory 1913)

# 3 Kernel convergence

Let $f_n : \mathbb{D} \to \Omega_n$, $n \in \mathbb{N}$ be conformal maps with suitable normalization. Can one characterize when $\{f_n\}$ converges locally uniformly on $\mathbb{D}$ in term of the regions $\Omega_n$? Yes! Answer related to *kernel convergence* of the sequence $\{\Omega_n\}$.

**Definition 3.1.** Let $\{\Omega_n\}$ be a sequence of regions in $\mathbb{C}$ and $w_0 \in \Omega_n$ for all $n \in \mathbb{N}$ ($w_0$ the *base point*). The kernel $\operatorname{Kern}_{w_0}$ w.r.t. $w_0$ of $\{\Omega_n\}$ consists of

i) the point $w_0$,

ii) every point $w \in \mathbb{C}$ with the following property: there exists a region $U$ with $w_0, w \in U$ such that $U \subseteq \Omega_n$ for all sufficiently large $n$.

So one always has $w_0 \in \operatorname{Kern}_{w_0}$, and $\operatorname{Kern}_{w_0} = \{w_0\}$ is possible. If $\operatorname{Kern}_{w_0} \neq \{w_0\}$, then $\operatorname{Kern}_{w_0}$ is a region ( = the union of sets $U$ in ii)).

Let $\Omega = \{w_0\}$ or $\Omega \subseteq \mathbb{C}$ be a region with $w_0 \in \Omega$. We say that $\{\Omega_n\}$ *converges to* $\Omega$ *in the sense of kernel convergence* (w.r.t. the base point $w_0$), written by

$$\Omega_n \to \Omega, \qquad \text{(w.r.t. } w_0\text{)},$$

if every subsequence of $\{\Omega_n\}$ has kernel $\Omega$.

**Example 3.2.** Let $\Omega_n = \mathbb{C} \setminus ((-\infty, -1/n] \cup [1/n, +\infty))$, $\mathbb{H}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, and $\mathbb{H}_- = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$. Then $\bigcap \Omega_n = \mathbb{H}_+ \cup \{0\} \cup \mathbb{H}_-$. Suppose $w_0 \in \mathbb{H}_+ \cup \{0\} \cup \mathbb{H}_-$ is the base point, then

$$\operatorname{Kern}_{w_0} = \begin{cases} \mathbb{H}_+ & \\ \{0\} & \text{for} \\ \mathbb{H}_- & \end{cases} \quad \begin{matrix} w_0 \in \mathbb{H}_+, \\ w_0 = 0, \\ w_0 \in \mathbb{H}_-. \end{matrix}$$

Moreover,

$$\Omega_n \to \begin{cases} \mathbb{H}_+ & \\ \{0\} & \text{w.r.t.} \\ \mathbb{H}_- & \end{cases} \quad \begin{matrix} w_0 \in \mathbb{H}_+, \\ w_0 = 0, \\ w_0 \in \mathbb{H}_-. \end{matrix}$$

**Lemma 3.3.** *Let $w_0 \in \mathbb{C}$, $\{\Omega_n\}$ be a sequence of regions in $\mathbb{C}$ with $w_0 \in \Omega_n$ for all $n \in \mathbb{N}$.*

a) *If $\{\Omega_n\}$ is increasing, i.e., $\Omega_n \subseteq \Omega_{n+1}$ for all $n \in \mathbb{N}$, then $\mathrm{Kern}_{w_0} = \Omega_\infty := \bigcup_{n\in\mathbb{N}} \Omega_n$, and $\Omega_n \to \Omega_\infty$ w.r.t. $w_0$.*

b) *If $\{\Omega_n\}$ is decreasing, i.e., $\Omega_n \supseteq \Omega_{n+1}$ for all $n \in \mathbb{N}$, let $\Omega_\infty$ be the connected component of the interior of $\bigcap_{n\in\mathbb{N}} \Omega_n$ containing $w_0$ if $w_0 \in \mathrm{int} \bigcap_{n\in\mathbb{N}} \Omega_n$ and $\Omega_\infty = \{w_0\}$ if not. Then $\mathrm{Kern}_{w_0} = \Omega_\infty$ and $\Omega_n \to \Omega_\infty$ w.r.t. $w_0$.*

*Proof.* a) $\mathrm{Kern}_{w_0} \subseteq \Omega_\infty$: clear.

$\Omega_\infty \subseteq \mathrm{Kern}_{w_0}$: if $w_0 \in \Omega_\infty$, then $w_0 \in \Omega_n$ for some $n \in \mathbb{N}$. Take $U = \Omega_n$ in Definition 3.1, so $w_0 \in \mathrm{Kern}_{w_0}$.

$\Omega_n \to \Omega_\infty$ because kernel (= union) does not change by passing to subsequences.

b) $\mathrm{Kern}_{w_0} \subseteq \bigcap_{n\in\mathbb{N}} \Omega_n$ is $\{w_0\}$ or a region containing $w_0$, so $\mathrm{Kern}_{w_0} \subseteq \Omega_\infty$.

$\Omega_\infty \subseteq \mathrm{Kern}_{w_0}$: clear if $\Omega_{w_0} = \{w_0\}$. Otherwise, take $U = \Omega_\infty$ in Definition 3.1, so $\Omega_\infty = U \subseteq \mathrm{Kern}_{w_0}$.

$\Omega_n \to \Omega_\infty$ is clear because $\bigcap_{n\in\mathbb{N}} \Omega_n$ does not change by passing to subsequences. □

**Proposition 3.4.** *Let $f_n : \mathbb{D} \leftrightarrow \Omega_n$ be conformal maps such that $f_n(0) = w_0$ and $f_n'(0) > 0$. Suppose that $f_n \to f$ locally uniformly on $\mathbb{D}$. Then, for the kernel of $\{\Omega_n\}$ w.r.t. $w_0$, we have $\mathrm{Kern}_{w_0} = f(\mathbb{D})$.*

*Proof.* Note that $f$ is a constant ($\equiv w_0$) or a conformal map onto $\Omega = f(\mathbb{D})$ (Hurwitz), $f(0) = w_0$.

I. $f(\mathbb{D}) \subseteq \mathrm{Kern}_{w_0}$: Obvious if $f$ is a constant. Assume $f$ is not a constant. Let $w \in f(\mathbb{D})$ be arbitrary. There exists $r \in (0,1)$ such that $w \in U := f(B(0,r))$. $U$ is a region such that $w_0, w \in U$ (and so $w \in \mathrm{Kern}_{w_0}$).

**Claim.** $U \subseteq f_n(\mathbb{D}) = \Omega_n$ for large $n$.

Otherwise, there exists a sequence $\{n_k\}$ in $\mathbb{N}$ with $n_k \to \infty$ and points $w_k \in U$ such that $w_k \notin f_{n_k}(\mathbb{D})$. Since $\overline{U} \subseteq f(\overline{B}(0,r))$ is compact, so wlog we can assume that $w_k \to v \in \overline{U} \subseteq f(\overline{\mathbb{D}})$. Then $h_k := f_{n_k} - w_k$ is zero-free on $\mathbb{D}$, and $h_k \to f - v$ locally uniformly on $\mathbb{D}$. However $v \in \overline{U} \subseteq f(\mathbb{D})$, so $f - v$ is not zero-free. So $f - v \equiv 0$, equivalently $f \equiv v$ by Hurwitz. Contradiction!

II. $\mathrm{Kern}_{w_0} \subseteq f(\mathbb{D})$: $w_0 \in f(\mathbb{D})$. Let $w \in \mathrm{Kern}_{w_0}$, $w \neq w_0$ be arbitrary. Then there exists a region $U$ such that $w_0, w \in U$ and $U \subseteq \Omega_n$ for all large $n$, wlog for all $n$. Then $g_n := f_n^{-1}|_U : U \to \mathbb{D}$ be holomorphic. By Montel's theorem, there exists a subsequence that converges locally uniformly to a holomorphic function $g : U \to \mathbb{D}$. Note that $g_n(w_0) = 0$ which implies $g(w_0) = 0$, and $g(U) \subseteq \overline{\mathbb{D}}$. So $g(U) \subseteq \mathbb{D}$ by Maximum principle.

Let $z := g(w) \in \mathbb{D}$. Then $f_n \to f$ locally uniformly near $z$, and so

$$w = \lim_{n\to\infty} f_n(g_n(w)) = f(z) \in f(\mathbb{D}).$$

A combination of I and II gives the proposition. □

**Theorem 3.5. (Main theorem about kernel convergence)** *Let $f_n : \mathbb{D} \leftrightarrow \Omega_n$ be conformal maps such that $f_n(0) = w_0$, $f_n'(0) > 0$ for $n \in \mathbb{N}$. Then*

i) $\Omega_n \to \{w_0\}$ *(w.r.t. $w_0$) iff $f_n \to$ const. $= w_0$ locally uniformly on $\mathbb{D}$ iff $f_n'(0) \to 0$.*

ii) $\Omega_n \to \Omega$, *where $\Omega \subseteq \mathbb{C}$ is a region in $\mathbb{C}$ with $w_0 \in \Omega$ and $\Omega \neq \mathbb{C}$ iff $f_n \to f \not\equiv$ const. locally uniformly on $\mathbb{D}$.*

iii) $\Omega_n \to \mathbb{C}$ *iff $f_n \to \infty$ locally uniformly on $\mathbb{D} \setminus \{0\}$ iff $f_n'(0) \to \infty$.*

*In particular, $\Omega_n \to \Omega \neq \mathbb{C}$ iff $\{f_n\}$ converges locally uniformly on $\mathbb{D}$.*

*Proof.* By Koebe's distortion theorem

$$|f_n'(0)|\frac{|z|}{(1+|z|)^2} \le |f_n(z) - w_0| \le |f_n'(0)|\frac{|z|}{(1-|z|)^2}, \tag{6}$$

and

$$B\Big(w_0, \frac{1}{4}|f_n'(0)|\Big) \subseteq \Omega_n = f_n(\mathbb{D}). \tag{7}$$

iii) First, $\Omega_n \to \mathbb{C} \Longrightarrow f_n'(0) \to \infty$. If not, then $\{f_n'(0)\}$ has a bounded subsequence, wlog, $\{f_n'(0)\}$ itself is bounded. By (6), $\{f_n\}$ is locally uniformly bounded on $\mathbb{D}$. By Montel's theorem, a subsequence of $\{f_n\}$ converges locally uniformly on $\mathbb{D}$, wlog, $f_n \to f$ locally uniformly. By Proposition 3.4, $\Omega_n = f_n(\mathbb{D}) \to f(\mathbb{D})$ w.r.t. $w_0$, but $f(\mathbb{D}) \neq \mathbb{C}$ (by Liouville). Contradiction!

Now, $f_n'(0) \to \infty \Longleftrightarrow f_n \to \infty$ locally uniformly on $\mathbb{D}$ by (6); and $f_n'(0) \to \infty \Longrightarrow \Omega_n \to \mathbb{C}$ by (7).

i) + ii) Suppose $\Omega_n \to \Omega \neq \mathbb{C}$ (possibly $\Omega = \{w_0\}$). Then by iii), $\{f_n'(0)\}$ has no subsequence $\{n_k\}$ with $f_{n_k}'(0) \to \infty$, and so $\{f_n'(0)\}$ is bounded. By (6), $\{f_n\}$ is locally uniformly bounded, and so a normal family by Montel. To show that $\{f_n\}$ converges locally uniformly on $\mathbb{D}$ it suffices that any two subsequential limits $g, h$ of $\{f_n\}$ agree. By Proposition 3.4,

$$g(\mathbb{D}) = \mathrm{Kern}_{w_0} = \Omega = h(\mathbb{D}).$$

So if $\Omega = \{w_0\}$, then $g = h \equiv w_0$, and $f_n \to w_0$ locally uniformly. This shows that $\Omega_n \to \{w_0\} \Longrightarrow f_n \to w_0$ locally uniformly.

If $\Omega \neq \{w_0\}$, then $g, h$ are conformal maps onto $\Omega$ by Hurwitz. We have $g(0) = h(0) = w_0$, and $g', h'$ are the subsequential limits of $\{f_n'\}$ by Weierstrass. So $g'(0), h'(0) > 0$. By uniqueness part of the Riemann mapping theorem, $g \equiv h$. This shows that $\Omega_n \to \Omega \neq \{w_0\}, \mathbb{C} \Longrightarrow f_n \to f$ locally uniformly, where $f$ is the unique conformal map with $\Omega = f(\mathbb{D})$, $f(0) = w_0$, $f'(0) > 0$.

Conversely,

i) $f_n \to w_0$ locally uniformly $\Longleftrightarrow f_n'(0) \to 0$ by (6) $\Longrightarrow \Omega_n \to \{w_0\}$ by Proposition 3.4.

ii) $f_n \to f \not\equiv$ const. $\Longrightarrow \Omega_n \to \Omega = f(\mathbb{D})$ by Proposition 3.4, so $f$ is a conformal map onto $f(\mathbb{D}) = \Omega \neq \mathbb{C}$. □

# 4 Loewner chains and the Loewner-Kufarev equation

**4.1. Loewner chains** (whole plane version)

Let $I = [a, \infty]$, $w_0$ be a base point, $\Omega_t$ be simply connected regions with $w_0 \in \Omega_t$ for $t \in I$ such that

i) $\Omega_\infty = \mathbb{C}$ ($\Omega_a = \{w_0\}$ is allowed as degenerate case),

ii) $\Omega_s \subsetneq \Omega_t$ for $s, t \in I$, $s < t$.

We say that the family $\{\Omega_t\}$ is a *(geometric) Loewner chain* if $\Omega_t$ is continuous in $t$ in the sense of kernel convergence w.r.t. $w_0$, i.e., $\Omega_{t_n} \to \Omega_t$ whenever $t_n \in I \to t \in I$.

For $t \in I$, let $f_t : \mathbb{D} \leftrightarrow \Omega_t$ be the unique conformal map with $f_t(0) = w_0$, $f_t'(0) > 0$ ($f_\infty$ is left undefined and $f_a = w_0$ if $\Omega_a = \{w_0\}$). Then $\{f_t\}$ is called an *(analytic) Loewner chain* if $f_t$ is continuous in $t$ w.r.t. locally uniform convergence on $\mathbb{D}$, i.e., $f_{t_n} \to f_t$ locally uniformly on $\mathbb{D}$ whenever $t_n \to t$. (It is understood that this means $f_{t_n}'(0) \to \infty$ if $t_n \to \infty$. No problem if $\Omega_a = \{w_0\}$ and $f_a = w_0$!)

The Loewner chain is normalized if $f_t'(0) = e^t$ for $t \in I$.

**Remark 4.2.** a) $\{\Omega_t\}$ continuous in $t$ if and only if $\{f_t\}$ continuous in $t$ (by Theorem 3.5).

b) For continuity of $\{f_t\}$, it is enough to check *left* and *right* continuity, i.e., that $f_{t_n} \to f_t$ locally uniformly on $\mathbb{D}$ whenever $t_n$ is a monotone sequence in $I$ (decreasing or increasing) with $t_n \to t$ (because every sequence has a monotone subsequence).

c) By a) and b), for continuity of $\{\Omega_t\}$, one only has to check that $\Omega_{t_n} \to \Omega_t$ whenever $t_n$ is a monotone sequence in $I$ with $t_n \to t$. By Lemma 3.3, this is equivalent to the following two conditions:

(i) $\Omega_t = \bigcup_{s<t} \Omega_s$ for $t \in I$, and

(ii) $\Omega_t = \{w_0\} \cup$ the connected component of interior of $\bigcap_{t<r} \Omega_r$ that contains $w_0$ for $t \in I$.

Note that if

(ii') $\Omega_t =$ interior of $\bigcap_{t<r} \Omega_r$, then (ii) is true.

d) Continuity of $\{\Omega_t\}$ is independent of $w_0 \in \bigcap \Omega_t = \Omega_a$. Indeed, (i) in a) is independent of $w_0$. Let $w_0, w_1 \in \bigcap \Omega_t$. Then $w_0, w_1 \in \Omega_t \subseteq$ interior of $\bigcap_{t<r} \Omega_r =: \tilde{\Omega}_t$. So $w_0, w_1$ lie in the same connected component of $\tilde{\Omega}_t$. This shows that (ii) true for $w_0$ iff true for $w_1$.

**Example 4.3.** (Loewner chain generated by slits)

Let $\gamma : [a, \infty] \to \hat{\mathbb{C}}$ be a simple path ending at $\infty$ (called it "slit"), i.e., $\gamma : [a, \infty] \to \hat{\mathbb{C}}$ be a continuous injective map with $\gamma(\infty) = \infty$. Let $\Omega_t = \mathbb{C} \setminus \gamma([t, \infty])$ for $t \in [a, \infty]$, $w_0 \in \mathbb{C} \setminus \gamma([a, \infty])$ (or $w_0 = \gamma(a)$, in this case $\Omega_a = \{w_0\}$). Then $\Omega_t$ is a simply connected region (the complement of an arc in $\hat{\mathbb{C}}$ has only one component!). $\Omega_s \subsetneq \Omega_t$ if $s < t$, because $\gamma([s, \infty]) \supsetneq \gamma([t, \infty])$.

For continuity,

(i) $\bigcup_{s<t} \mathbb{C} \setminus \gamma([s, \infty]) = \mathbb{C} \setminus \bigcap_{s<t} \gamma([s, \infty]) = \mathbb{C} \setminus \gamma(\bigcap_{s<t}[s, \infty])$ (by continuity of $\gamma$) $= \mathbb{C} \setminus \gamma([t, \infty]) = \Omega_t$.

(ii') $\bigcap_{t<r} \mathbb{C} \setminus \gamma([r, \infty]) = \mathbb{C} \setminus \bigcup_{t<r} \gamma([r, \infty]) = \mathbb{C} \setminus \gamma(\bigcup_{t<r}[r, \infty]) = \mathbb{C} \setminus \gamma((t, \infty]) = \Omega_t \cup \gamma(t)$. So $\mathrm{int}(\bigcap_{t<r} \mathbb{C} \setminus \gamma([r, \infty]) = \Omega_t$. (If $t = a$, $w_0 = \gamma(a)$, $\Omega_a = \{w_0\}$, then $\tilde{\Omega} := \mathrm{int}(\bigcap_{t<r} \mathbb{C} \setminus \gamma([r, \infty])) = \mathbb{C} \setminus \gamma([a, \infty])$. So the component of $\tilde{\Omega}$ containing $w_0 = \varnothing$, and (ii) true for $t = a$.)

**Example 4.4.** Let $\Omega$ be a bounded Jordan region. Then there exists a Loewner chain $\{\Omega_t\}_{t \in [1, \infty]}$ such that $\Omega_1 = \Omega$ ($w_0 \in \Omega$).

*Proof.* Let $\tilde{\Omega}$ be the exterior of the Jordan curve $\partial\Omega$ in $\hat{\mathbb{C}}$. Then there exists a conformal map $f : \tilde{\mathbb{D}} \to \tilde{\Omega}$ with $f(\infty) = \infty$. It has a homeomorphic extension $f : \overline{\tilde{\mathbb{D}}} \to \overline{\tilde{\Omega}}$.

For $t \in [1, \infty)$, let $\Omega_t$ be the inside of the Jordan curve $f(\{z \in \mathbb{C} : |z| = t\})$ and $\Omega_\infty = \mathbb{C}$. Then $\{\Omega_t\}_{t \in [1, \infty]}$ is a Loewner chain with $\Omega_1 = \Omega$.

$\Omega_1 = \Omega$ is clear. $\Omega_t$ is strictly increasing. Indeed,

$$\Omega_t = \hat{\mathbb{C}} \setminus f(\tilde{\mathbb{D}}) \cup f(\{z \in \mathbb{C} : 1 < |z| < t\}), \qquad \text{for } 1 < t < \infty.$$

(shown as in the proof of Area Theorem.)

Continuity:

For $1 \leq t < \infty$,

(i) $\bigcup_{s<t} \Omega_s = \hat{\mathbb{C}} \setminus f(\tilde{\mathbb{D}}) \cup f(\{z \in \mathbb{C} : 1 < |z| < t\}) = \Omega_t$.

(ii') $\bigcap_{s\leq t} \Omega_s = \hat{\mathbb{C}} \setminus f(\tilde{\mathbb{D}}) \cup f(\{z \in \mathbb{C} : 1 < |z| \leq t\}) = \Omega_t \cup \partial\Omega_t = \overline{\Omega}_t$. Since $\Omega_t$ is a Jordan region, $\text{int}(\overline{\Omega}) = \Omega_t$.

For $t = \infty$, $\bigcup_{s<\infty} \Omega_s = \hat{\mathbb{C}} \setminus f(\tilde{\mathbb{D}}) \cup f(\{z \in \mathbb{C} : 1 < |z| < \infty\}) = \overline{\Omega}_t \cup \tilde{\Omega}_t \setminus \{\infty\} = \mathbb{C}$. □

### 4.5. The associated semi-group

Let $f, g : \mathbb{D} \to \mathbb{C}$ be two holomorphic maps. $f$ is *subordinate* to $g$, written by $f \prec g$, if there exists a holomorphic map $\varphi : \mathbb{D} \to \mathbb{D}$ with $\varphi(0) = 0$ such that $f = g \circ \varphi$ (then $f(0) = g(0)$, and $|f'(0)| \leq |g'(0)|$, because $|\varphi'(0)| \leq 1$ by Schwarz's Lemma).

Let $\{f_t\}_{t\in[a,\infty]}$ be a Loewner chain. For $a \leq s \leq t < \infty$, $\Omega_s \subseteq \Omega_t$, so $f_t^{-1}$ is defined on $\Omega_s$. Let $\varphi_{s,t} := f_t^{-1} \circ f_s : \mathbb{D} \to \mathbb{D}$. Then $\varphi_{s,t}$ is a conformal map onto its image. $\varphi_{s,t}(\mathbb{D}) \subseteq \mathbb{D}$ and $\varphi_{s,t}(0) = 0$. We have

$$f_s = f_t \circ \varphi_{s,t}, \qquad a \leq s \leq t < \infty, \tag{8}$$

$$\varphi_{t,u} \circ \varphi_{s,t} = \varphi_{s,u}, \quad a \leq s \leq t \leq u < \infty, \quad \text{(semi-group property)}$$

$$\varphi_{t,t} = \text{id}_{\mathbb{D}}, \qquad a \leq t < \infty.$$

(8) shows that $f_s$ is subordinate to $f_t$ for $s < t$, so

$$f_s'(0) \leq f_t'(0), \qquad s < t.$$

Actually, we have strict inequality

$$f_s'(0) < f_t'(0). \qquad s < t.$$

Otherwise, $f_t'(0) = f_s'(0) = f_t'(0) \cdot \varphi_{s,t}'(0)$, so $\varphi_{s,t}'(0) = 1$. By Schwarz's Lemma, $\varphi_{s,t} = \text{id}_{\mathbb{D}}$, and $f_t = f_s$, $\Omega_t = f_t(\mathbb{D}) = f_s(\mathbb{D}) = \Omega_s$. A contradiction.

### 4.6. Heuristics for the Loewner equation

A family of maps $\varphi_{s,t}$ with the semi-group property is generated by a time-dependant vector field.

Assume $\varphi_{s,t}(z)$ is smooth in $s, t$, holomorphic in $z$. Define

$$V(z, s) = \frac{\partial \varphi_{s,t}}{\partial t}(z)\Big|_{t=s} = \lim_{\delta\to 0^+} \frac{\varphi_{s,s+\delta}(z) - z}{\delta}.$$

$V(z, s)$ forms a time-dependent vector field. Note that $\varphi_{s,s} = \text{id}_{\mathbb{D}}$, $\varphi_{s,s+\delta}(z) \sim z + \delta V(z, s)$. We have

$$\frac{\partial \varphi_{s,t}}{\partial t}(z) = \lim_{\delta\to 0^+} \frac{\varphi_{s,t+\delta}(z) - \varphi_{s,t}(z)}{\delta} = \lim_{\delta\to 0^+} \frac{\varphi_{t,t+\delta}(\varphi_{s,t}(z)) - \varphi_{s,t}(z)}{\delta} = V(\varphi_{s,t}(z), t).$$

So the semi-group $\varphi_{s,t}$ satisfies the following equations

$$\frac{\partial \varphi_{s,t}}{\partial t}(z) = V(\varphi_{s,t}(z), t), \qquad t > s,$$
$$\frac{\partial \varphi_{s,t}}{\partial t}(z)\Big|_{t=s} = V(z, s).$$

Let $\gamma : [s, u] \to \mathbb{C}$ be a $C^1$-smooth curve satisfying

$$\gamma(s) = z, \quad \dot{\gamma}(t) = V(\gamma(t), t), \quad t \in [s, u].$$

Then $\gamma$ is an integral curve of the vector field $V$. So $t \to \varphi_{s,t}(z)$ is an integral curve of $V$. In fact, $z$ at time $s \mapsto \gamma(t)$ at time $t$ is a map $\varphi_{s,t}(z)$ (map from time $s$ to $t$).

*What can we say about $V(z, s)$ if $\varphi_{s,t}$ comes from Loewner chain?*

By Schwarz's Lemma, $\varphi_{t,t+\delta}(z) \in \overline{B}(0, |z|)$. So $\mathrm{Re}((\varphi_{t,t+\delta}(z) - z)/z) \leq 0$, and

$$\mathrm{Re}\,\frac{V(z,t)}{z} = \mathrm{Re}\lim_{\delta \to 0^+} \frac{\varphi_{t,t+\delta}(z) - z}{\delta z} \leq 0.$$

So $V(z, t)$ can be written as

$$V(z,t) = -zp(z,t),$$

where $p(z, t)$ is holomorphic in $z$ and $\mathrm{Re}\, p(z, t) \geq 0$ for $z \in \mathbb{D}$.

Let $\{f_t\}$ be a Loewner chain and $f(z, t) := f_t(z)$. Assume that $f(z, t)$ is smooth in $t$. Denote

$$f_t'(z) = \frac{\partial f}{\partial z}(z,t), \qquad \dot{f}_t(z) = \frac{\partial f}{\partial t}(z,t).$$

For $\varepsilon > 0$,

$$f_t(z) = f_{t+\varepsilon} \circ \varphi_{t,t+\varepsilon}(z) = f(\varphi_{t,t+\varepsilon}(z), t+\varepsilon).$$

So

$$\begin{aligned} 0 = \frac{\partial f_t(z)}{\partial \varepsilon}\Big|_{\varepsilon=0} &= \frac{\partial}{\partial \varepsilon} f(\varphi_{t,t+\varepsilon}(z), t+\varepsilon)\Big|_{\varepsilon=0} \\ &= f_t'(z)\frac{\partial \varphi_{t,t+\varepsilon}(z)}{\partial \varepsilon}(z)\Big|_{\varepsilon=0} + \dot{f}_t(z) \\ &= f_t'(z)V(z,t) + \dot{f}_t(z) \\ &= -\,zp(z,t)f_t'(z) + \dot{f}_t(z). \end{aligned}$$

The equation

$$\dot{f}_t(z) = zp(z,t)f_t'(z), \tag{9}$$

i.e.

$$\frac{\partial f}{\partial t}(z,t) = zp(z,t)\frac{\partial f}{\partial z}(z,t)$$

is called the *Loewner-Kufarev equation.*

*Have we accomplished anything?*

Wlog, assume $f(0, t) = w_0 \equiv 0$, $f_0 \in \mathcal{S}$ (i.e.$a_1(0) = 1$). Let

$$\begin{aligned} f(z,t) &= a_1(t)z + a_2(t)z^2 + \cdots. \\ \dot{f}(z,t) &= \dot{a}_1(t)z + \dot{a}_2(t)z^2 + \cdots, \\ f'(z,t) &= a_1(t) + 2a_2(t)z + \cdots, \\ p(z,t) &= c_0(t) + c_1(t)z + \cdots. \end{aligned}$$

Then

$$(\dot a_1(t)z + \dot a_2(t)z^2 + \cdots) = z(c_0(t) + c_1(t)z + \cdots)(a_1(t) + 2a_2(t)z + \cdots)$$
$$= c_0a_1z + (c_1a_1 + 2c_0a_2)z^2 + \cdots.$$

Comparing coefficients, we get

$$\dot a_1 = c_0a_1, \qquad \dot a_2 = c_1a_1 + 2c_0a_2.$$

Making a change of time parametrization, we can assume $\dot a_1 = a_1$, so

$$c_0 = 1 \qquad \text{and} \qquad a_1(t) = e^t.$$

Now

$$\dot a_2 - 2a_2 = c_1e^t.$$

So

$$a_2(t) = C(t)e^{2t}, \qquad \text{where} \quad C(t) = \int_0^t c_1(s)e^{-s}ds.$$

Since $e^{-t}f_t \in \mathcal{S}$, we have $|a_2(t)e^{-t}|$ is bounded. So

$$C(\infty) = \lim_{t\to\infty} C(t) = \lim_{t\to\infty} a_2(t)e^{-2t} = 0,$$

$$-C(t) = C(\infty) - C(t) = \int_t^\infty c_1(s)e^{-s}ds.$$

So

$$a_2(t) = -e^{2t}\int_t^\infty c_1(s)e^{-s}ds, \quad \text{and} \quad a_2(0) = -\int_0^\infty c_1(t)e^{-t}dt.$$

Note that if $f(z) = 1 + c_1z + c_2z^2 + \cdots$ holomorphic in $\mathbb{D}$, and $\operatorname{Re} f(z) \geq 0$, then $|c_2| \leq 2$ by Schwarz's Lemma. So $|c_1(0)| \leq 2$ and

$$|a_2(0)| \leq 2\int_0^\infty e^{-t}dt \leq 2.$$

**Lemma 4.7.** *Let $\{f_t\}_{t\in I}$, $I = [a,\infty]$, be an analytic Loewner chain. Then there exist $\tilde a \in [-\infty, +\infty)$, a strictly increasing homeomorphism $\alpha : \tilde I := [\tilde a, \infty] \to I$, and a Loewner chain $\{\tilde f_t\}_{t\in\tilde I}$ such that*

i) $\tilde f_t'(0) = e^t$ *for* $t \in \tilde I \setminus \{-\infty, \infty\}$,

ii) $\tilde f_t = f_{\alpha(t)}$.

(So by a homeomorphic change of time parametrization, one can normalize an analytic Loewner chain.)

*Proof.* Define

$$\beta(t) = \begin{cases} f_t'(t) & \text{for} \quad t \in I\setminus\{\infty\} \\ \infty & \phantom{\text{for}} \quad t = \infty \end{cases}.$$

Then

i) $\beta$ is strictly increasing (see 4.5).

ii) $\beta$ is continuous:

Let $\{t_n\}$ be a sequence in $I$ such that $t_n \to t_\infty \in I$. Then if $t_\infty = \infty$, $\beta(t_n) = f'_{t_n}(0) \to \infty = \beta(\infty)$ by the definition of Loewner chain; if $t_\infty \neq \infty$, $f_{t_n} \to f_{t_\infty}$ locally uniformly on $\mathbb{D}$; so $\beta(t_n) = f'_{t_n}(0) \to f'_{t_\infty}(0) = \beta(t_\infty)$ by Weierstrass theorem.

By i) + ii), $\beta$ is a homeomorphism onto its image $\tilde{I} := \beta(I) = [b, \infty] \subseteq [0, \infty]$. Let $\tilde{a} := \log b \in [-\infty, \infty)$, and $\alpha(t) := \beta^{-1}(e^t)$, $t \in [\tilde{a}, \infty]$ $(e^{-\infty} = 0, e^{\infty} = \infty)$. Then $\alpha$ is a strictly increasing homeomorphism from $\tilde{I} := [\tilde{a}, \infty]$ onto $I = [a, \infty]$.

$$\tilde{I} \xleftrightarrow{\exp} [b, \infty] \xleftrightarrow{\beta^{-1}} [a, \infty].$$

Define $\tilde{f}_t := f_{\alpha(t)}$. Then $\{\tilde{f}_t\}_{t\in\tilde{I}}$ is a Loewner chain (obvious), and

$$\tilde{f}'_t(0) = f'_{\alpha(t)}(0) = \beta(\alpha(t)) = e^t, \quad \text{for } t \in \tilde{I}. \qquad \square$$

From now on, all analytic Loewner chain $\{t_t\}_{t\in I}$ are normalized, i.e., $f'_t(0) = e^t$ for $t \in I$.

**Theorem 4.8. (Vitali's theorem on induced convergence)** *Let $\Omega \subseteq \mathbb{C}$ be a region, $\mathcal{F}$ be a normal family of holomorphic functions on $\Omega$, and $\{f_n\}$ be a sequence in $\mathcal{F}$. Suppose there exists a sequence $\{z_k\}$ of points in $\Omega$ such that*

i) *$\{f_n(z_k)\}$ converges for all $k \in N$,*

ii) *$\{z_k\}$ has a limit point in $\Omega$.*

*Then $\{f_n\}$ converges locally uniformly on $\Omega$ (to a holomorphic limit function $f$).*

*Proof.* There exists a subsequential limit $f \in H(\Omega)$ of $\{f_n\}$ (w.r.t. locally uniform convergence on $\Omega$).

**Claim.** $f_n \to f$ locally uniformly on $\Omega$.

We prove it by contradiction. If not, then there exist $\varepsilon_0 > 0$ ("bad $\varepsilon$"), a compact set $K \subseteq \Omega$, a sequence $n_l \in \mathbb{N}$ with $n_l \to \infty$, and points $u_l \in K$ such that

$$|f_{n_l}(u_l) - f(u_l)| \geq \varepsilon_0.$$

Let $g_l$ denote $f_{n_l}$. Then $\{g_l\}$ is a sequence in $\mathcal{F}$, so it has a convergent subsequence, wlog, $g_l \to g$ locally uniformly on $\Omega$. Also, wlog, $u_l \to u_\infty \in K$. Since $\{f_n(z_k)\}$ converges for each $k \in \mathbb{N}$, we have $g(z_k) = f(z_k)$. Since $\{z_k\}$ has a limit point in $\Omega$, $g \equiv f$ by the Uniqueness Theorem. So

$$0 < \varepsilon_0 \leq \lim_{l\to\infty} |g_l(u_l) - f(u_l)| = |g(u_\infty) - f(u_\infty)| = 0.$$

Contradiction! $\square$

**Theorem 4.9. (Holomorphic functions with positive real part)** *Let $\mathcal{P} = \{p \in H(\mathbb{D}) : p(0) = 1, \operatorname{Re} p \geq 0 \text{ on } \mathbb{D}\}$. Then the following statements are true.*

i) $|p(z)| \leq \dfrac{1+|z|}{1-|z|}$ *for all $p \in \mathcal{P}$ and $z \in \mathbb{D}$.*

ii) *$\mathcal{P}$ is a normal family, and it is closed w.r.t. locally uniform convergence, i.e., if $\{p_n\}$ is a sequence in $\mathcal{P}$ and $p_n \to p$ locally uniformly on $\mathbb{D}$, then $p \in \mathcal{P}$.*

iii) *If $p \in \mathcal{P}$, then there exists a unique Borel probability measure $\mu$ on $\partial\mathbb{D}$ such that*

$$p(z) = \int_{\partial\mathbb{D}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta) \qquad \textit{for} \quad z \in \mathbb{D}.$$

*(Herglotz representation). Conversely, every function of this type belongs to $\mathcal{P}$.*

*If $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is the Taylor expansion of $p$, then*

$$c_n = 2\int_{\partial\mathbb{D}} \zeta^{-n} d\mu(\zeta) = 2\int_0^{2\pi} e^{-in\theta} d\mu(e^{i\theta}) \qquad \textit{for} \quad n \in \mathbb{N}.$$

iv) *Let $p(z) = 1 + c_1 z + c_2 z^2 + \cdots \in \mathcal{P}$. Then $|c_n| \le 2$ and $(\operatorname{Re} c_1)^2 \le 2 + \operatorname{Re} c_2$.*

*Proof.* Note that $\operatorname{Re} p > 0$ for $p \in \mathcal{P}$ by the minimal principle for holomorphic functions.

i) It can be easily obtained by Schwarz's Lemma (details filled later).

ii) By i), $\mathcal{P}$ is locally uniformly bounded. The remains obtained by the Montel theorem and the Weierstrass theorem.

iii) Let $p \in \mathcal{P}$. For fixed $r \in (0,1)$, define $p_r(z) = p(rz)$. The $p_r \in H(\mathbb{D})$ and $p_r$ has a continuous extension to $\overline{\mathbb{D}}$. Hence, by the Schwarz formula

$$p_r(z) = \operatorname{Im} p_r(0) + \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \operatorname{Re} p_r(e^{it})dt = \int_{\partial\mathbb{D}} \frac{\zeta+z}{\zeta-z} d\mu_r(\zeta),$$

where

$$d\mu_r(\zeta) = d\mu_r(e^{it}) = \frac{1}{2\pi}\operatorname{Re} p_r(e^{it})dt = \frac{1}{2\pi}\operatorname{Re} p(r\zeta)dt.$$

$\mu_r$ is a positive Borel measure on $\partial\mathbb{D}$, and

$$\mu_r(\partial\mathbb{D}) = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re} p(re^{it})dt = \operatorname{Re} p(0) = 1.$$

So $\mu_r$ is a positive Borel probability measure on $\partial\mathbb{D}$.

By Banach-Alaoglu theorem, there exists a sequence $r_n \in (0,1)$ with $r_n \to 1$ such that $\mu_n := \mu_{r_n} \to \mu$ w.r.t. the weak-$*$ topology on $C(\partial\mathbb{D})^* = \{\nu : \text{complex Borel measure on } \partial\mathbb{D}\}$, i.e.,

$$\int_{\partial\mathbb{D}} u d\mu_n \to \int_{\partial\mathbb{D}} u d\mu \qquad \text{for all} \quad u \in C(\partial\mathbb{D}).$$

$\mu$ is also a probability measure. For fixed $z \in \mathbb{D}$, we have

$$\begin{aligned} p(z) &= \lim_{n\to\infty} p(r_n z) = \lim_{n\to\infty} p_{r_n}(z) \\ &= \lim_{n\to\infty} \int_{\partial\mathbb{D}} \frac{\zeta+z}{\zeta-z} d\mu_n(\zeta) = \int_{\partial\mathbb{D}} \frac{\zeta+z}{\zeta-z} d\mu(\zeta). \end{aligned}$$

This shows the existence of the Herglotz representation.

Uniqueness and converse will be the homework assignments!

For fixed $z \in \mathbb{D}$ and $\zeta \in \partial\mathbb{D}$, we have

$$\frac{\zeta+z}{\zeta-z} = \frac{1+z/\zeta}{1-z/\zeta} = 1 + 2\sum_{n=1}^{\infty} z^n \zeta^{-n},$$

converges uniformly in $\zeta$. So we can integral term-by-term and conclude

$$\begin{aligned}p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n &= \int_{\partial\mathbb{D}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta) \\ &= \int_{\partial\mathbb{D}} \Big(1 + 2\sum_{n=1}^{\infty} z^n \zeta^{-n}\Big) d\mu(\zeta) \\ &= 1 + 2\sum_{n=1}^{\infty} \Big(\int_{\partial\mathbb{D}} \zeta^{-n} d\mu(\zeta)\Big) z^n, \qquad \text{for all} \quad z \in \mathbb{D}.\end{aligned}$$

Comparing coefficients, we can obtain

$$c_n = 2\int_{\partial\mathbb{D}} \zeta^{-n} d\mu(\zeta) \qquad \text{for} \quad n \in \mathbb{N}.$$

iv) In particular,

$$|c_n| = 2\Big|\int_{\partial\mathbb{D}} \zeta^{-n} d\mu(\zeta)\Big| \le 2\int_{\partial\mathbb{D}} |\zeta^{-n}| d\mu(\zeta) = 2.$$

Here we have used $\zeta = e^{it}$. So

$$\operatorname{Re} c_1 = 2\int_{\partial\mathbb{D}} \operatorname{Re}(e^{-it}) d\mu(\zeta) = 2\int_{\partial\mathbb{D}} (\cos t) d\mu(\zeta), \quad \text{and} \quad \operatorname{Re} c_2 = 2\int_{\partial\mathbb{D}} (\cos 2t) d\mu(\zeta).$$

So

$$\begin{aligned}(\operatorname{Re} c_1)^2 = 4\Big(\int_{\partial\mathbb{D}} (\cos t) d\mu(\zeta)\Big)^2 &\le 4\int_{\partial\mathbb{D}} (\cos^2 t) d\mu(\zeta) \qquad \text{(Cauchy-Schwarz)} \\ &= 4\int_{\partial\mathbb{D}} \frac{1 + \cos 2t}{2} d\mu(\zeta) = 2 + 2\operatorname{Re} c_2.\end{aligned}$$

□

**Lemma 4.10.** *Let $\{f_t\}_{t\in[a,\infty]}$ be a normalized Loewner chain, $\varphi_{s,t} = f_t^{-1} \circ f_s$ for $s \le t$ on $I = [a, \infty]$. Then for fixed $z \in \mathbb{D}$,*

i) $|\varphi_{s,t}(z) - z| \le |t - s|\dfrac{2|z|}{1 - |z|}, \quad a \le s \le t < \infty,$

ii) $|f_t(z) - f_s(z)| \le e^t|t - s|\dfrac{4|z|}{(1 - |z|)^4}, \quad a \le s \le t < \infty,$

iii) $|\varphi_{s,u}(z) - \varphi_{t,u}(z)| \le |t - s|\dfrac{2|z|}{(1 - |z|)^2}, \quad a \le s \le t \le u < \infty,$

iv) $|\varphi_{s,t}(z) - \varphi_{s,u}(z)| \le |u - t|\dfrac{2|z|}{1 - |z|}, \quad a \le s \le t \le u < \infty.$

*So the following functions are Lipschitz:*

$t \to f_t(z)$ *on* $[a, \infty)$, $z \in \mathbb{D}$ *fixed;*

$t \to \varphi_{s,t}(z)$ *on* $[s, \infty)$, $z \in \mathbb{D}$, $s \in [a, \infty)$ *fixed;*

$t \to \varphi_{t,u}(z)$ *on* $[a, u]$, $z \in \mathbb{D}$, $u \in [a, \infty)$ *fixed.*

*Moreover, the Lipschitz constants are uniform if the arguments and parameters are restricted to suitable subdomains. For example, for each $n \in \mathbb{N}$, there exists $L = L(n)$ such that $t \to f_t(z)$ is L-Lipschitz on $[a, n]$ for each $z \in \overline{B}(0, 1 - \frac{1}{n})$.*

*Proof.* Some estimates:

1. $|h(z_1) - h(z_2)| \le \max_{|u|\le r} |h'(u)||z_1 - z_2|$ for $h \in H(\mathbb{D})$, $z_1, z_2 \in \overline{B}(0,r)$, $0 < r < 1$.
2. $|e^u - e^v| \le |u - v|$, $u, v \in \mathbb{C}$, $\operatorname{Re} u, \operatorname{Re} v \le 0$.
3. $\varphi \in \operatorname{Aut}(\mathbb{D})$. Then

$$|\varphi'(z)| \le \frac{1 - |\varphi(z)|^2}{1 - |z|^2} \le \frac{1}{1 - |z|^2}, \quad z \in \mathbb{D}. \qquad \text{(Schwarz-Pick)}$$

i) From $s \le t$, $f_t \circ \varphi_{s,t} = f_s$, we have

$$f_t'(\varphi_{s,t}(0)) \cdot \varphi_{s,t}'(0) = f_s'(0).$$

By $\varphi_{s,t}(0) = 0$, $f_t'(0) = e^t$, $e^t \cdot \varphi_{s,t}'(0) = e^s$, so $\varphi_{s,t}'(0) = e^{s-t} \le 1$. Define

$$\Phi_{s,t}(z) = \log\left(\frac{z}{\varphi_{s,t}(z)}\right) = \log \frac{z}{e^{s-t}z + \cdots} = \log(e^{t-s} + \cdots) = (t - s) + \cdots . \tag{10}$$

Then $\Phi_{s,t}$ is holomorphic in $\mathbb{D}$ and $\Phi_{s,t}(0) = t - s$. Since $|z/\varphi(z)| \ge 1$, so $\operatorname{Re}\Phi_{s,t}(z) \ge 0$, and $\frac{1}{t-s}\Phi_{s,t} \in \mathcal{P}$. Hence, by Theorem 4.9,

$$|\Phi_{s,t}(z)| \le |t - s|\frac{1 + |z|}{1 - |z|} \le |t - s|\frac{2}{1 - |z|}.$$

From $\varphi_{s,t}(z) = z \cdot e^{-\Phi_{s,t}(z)}$, $\operatorname{Re}\Phi_{s,t}(z) \ge 0$, we have

$$|\varphi_{s,t}(z) - z| = |z||e^{-\Phi_{s,t}(z)} - e^0| \le |z||\Phi_{s,t}(z)| \le |t - s|\frac{2|z|}{1 - |z|}.$$

ii) $|f_t(z) - f_s(z)| = |f_t(z) - f_t(\varphi_{s,t}(z))| \le \max_{|u|\le|z|} |f_t'(u)||z - \varphi_{s,t}(z)|$, here we have used $|\varphi_{s,t}(z)| \le |z|$. By Koebe's and i),

$$|f_t(z) - f_s(z)| \le e^t\frac{1 + |z|}{(1 - |z|)^3} \cdot |t - s|\frac{2|z|}{1 - |z|} \le e^t|t - s|\frac{4|z|}{(1 - |z|)^4}.$$

iii) By Schwarz lemma and i),

$$\begin{aligned}|\varphi_{s,u}(z) - \varphi_{t,u}(z)| &= |\varphi_{t,u}(\varphi_{s,t}(z)) - \varphi_{t,u}(z)| \le \max_{|a|\le|z|} |\varphi_{t,u}'(a)| \cdot |\varphi_{s,t}(z) - z| \\ &\le \frac{1}{1 - |z|^2} \cdot |t - s|\frac{2|z|}{1 - |z|} \le |t - s|\frac{2|z|}{(1 - |z|)^2}.\end{aligned}$$

iv) By i) and $|\varphi_{s,t}(z)| \le |z|$,

$$\begin{aligned}|\varphi_{s,t}(z) - \varphi_{s,u}(z)| &= |\varphi_{s,t}(z) - \varphi_{t,u}(\varphi_{s,t}(z))| \\ &\le |u - t|\frac{2|w|}{1 - |w|} \le |u - t|\frac{2|z|}{1 - |z|}, \qquad \text{where } w = \varphi_{s,t}(z).\end{aligned}$$ □

**Definition 4.11.** Let $\Omega \subseteq \mathbb{C}$ be a region, $I \subseteq \mathbb{R}$ be an interval. $HL(\Omega \times I)$ is the set of all function $f : \Omega \times I \to \mathbb{C}$ satisfying

i) $f(\cdot, t)$ is holomorphic on $\Omega$ for all $t \in I$,

ii) $f(z, \cdot)$ is uniformly Lipschitz on compact set, i.e., whenever, $K \subseteq \Omega$ compact, $J \subseteq I$ compact interval, then there exists $L > 0$ such that $|f(z, s) - f(z, t)| \le L|s - t|$ for all $z \in K$ and all $s, t \in J$.

Lemma 4.10 shows that if $\{f_t\}$ is a normalized Loewner chain on $[a,\infty]$, then
$(z,t) \to f_t(z) \in HL(\mathbb{D},[a,\infty))$;
$(z,t) \to \varphi_{s,t}(z) \in HL(\mathbb{D},[s,\infty))$;
$(z,s) \to \varphi_{s,t}(z) \in HL(\mathbb{D},[a,t])$, where $\varphi_{s,t} = f_t^{-1} \circ f_s$.

**Proposition 4.12.** *Let $\Omega \subseteq \mathbb{C}$ be a region, $I \subseteq \mathbb{R}$ be an interval, $f \in HL(\Omega \times I)$. Then*
i) *$f$ is continuous on $\Omega \times I$.*
*There exists a set $E \subseteq I$ with $|E| = 0$ (the 1-dim Lebesgue measure) such that*
ii) *$\frac{\partial f}{\partial t}(z,t)$ exists for all $z \in \Omega$, $t \in I \setminus E$. Moreover, $\frac{\partial f}{\partial t}(z,t)$ is holomorphic on $\Omega$ for all $t \in I \setminus E$, $\frac{\partial f}{\partial t}$ is measurable and uniformly bounded on compact subsets, i.e., whenever $K \subseteq \Omega$ compact, $J \subseteq I$ compact interval, then there exists $M \geq 0$ such that $\left|\frac{\partial f}{\partial t}(z,t)\right| \leq M$ for all $z \in K$, $t \in J \setminus E$.*
iii) *$f$ is differentiable at each point $(z,t) \in \Omega \times I \setminus E$, more precisely,*

$$f(z',t') = f(z,t) + \frac{\partial f}{\partial z}(z,t)(z'-z) + \frac{\partial f}{\partial t}(z,t)(t'-t) + o(|z'-z| + |t'-t|)$$

*as $(z',t') \to (z,t)$.*
iv) *$\frac{\partial^n f}{\partial z^n} \in HL(\Omega \times I)$ for all $n \in \mathbb{N}$. Moreover,*

$$\frac{\partial}{\partial t}\Big(\frac{\partial^n f}{\partial z^n}\Big)(z,t) = \frac{\partial^n}{\partial z^n}\Big(\frac{\partial f}{\partial t}\Big)(z,t) \qquad \text{for all} \quad (z,t) \in \Omega \times I \setminus E. \tag{11}$$

v) *Let $z_0 \in \Omega$, and*

$$f(z,t) = \sum_{n=0}^{\infty} a_n(t)(z-z_0)^n$$

*be the Taylor expansion of $f(\cdot,t)$ at $z_0$. Then for each $n \in \mathbb{N}$, $a_n(t)$ is uniformly Lipschitz on compact interval $J \subseteq I$. Moreover, $\dot{a}_n(t) := \frac{da_n}{dt}(t)$ exists for all $t \in I \setminus E$, and for $t \in I \setminus E$, the function $\frac{\partial f}{\partial t}(\cdot,t)$ has the Taylor expansion*

$$\frac{\partial f}{\partial t}(z,t) = \sum_{n=0}^{\infty} \dot{a}_n(t)(z-z_0)^n. \tag{12}$$

*Proof.* i) $|f(z',t') - f(z,t)| \leq |f(z',t') - f(z',t)| + |f(z',t) - f(z,t)|$ is small if $|z'-z| + |t'-t|$ small, since $|f(z',t') - f(z',t)|$ is uniformly small and $|f(z',t) - f(z,t)$ is small.

ii) Pick a sequence $\{a_k\}$ in $\Omega$ of distinct points such that $\{a_k\}$ has a limit point in $\Omega$ (e.g. $a_k = a_0 + \delta/k$, $a_0 \in \Omega$, $\delta > 0$ small. Each function $t \mapsto f(a_k,t)$ is locally Lipschitz on $I$, and so differentiable a.e. on $I$. So there exists a set $E_k \subseteq I$ with $|E_k| = 0$ such that $\frac{\partial f}{\partial t}(a_k,t)$ exists for each $t \in I \setminus E_k$. Let $E = \bigcup_{k\in\mathbb{N}} E_k \cup \{\text{end points of } I\} \subseteq I$. Then $|E| = 0$.

**Claim.** $\frac{\partial f}{\partial t}(z,t)$ exists for all $(z,t) \in \Omega \times I \setminus E$.

It suffices to show that if $\{\delta_n\}$ is a sequence in $\mathbb{R}$ with $\delta_n \neq 0$ and $\delta_n \to 0$, then

$$\lim_{n\to\infty} \frac{f(z, t+\delta_n) - f(z,t)}{\delta_n} \tag{13}$$

exists (then the limit is independent of $\{\delta_n\}$).

Define

$$F_n(z') := \frac{f(z', t+\delta_n) - f(z', t)}{\delta_n} \qquad \text{for } z' \in \Omega.$$

Then $\{F_n\}$ is a sequence of holomorphic functions on $\Omega$ that are locally uniformly bounded on $\Omega$, and so form a normal family.

$$F_n(a_k) \to \frac{\partial f}{\partial t}(a_k, t) \qquad \text{as } n \to \infty$$

for each $k \in \mathbb{N}$. By Vitali's Theorem 4.8, $\{F_n(z')\}$ converges for each $z' \in \Omega$, and so also for $z' = z$; so the limit (13) exists. So $\frac{\partial f}{\partial t}(z,t)$ exists for all $(z,t) \in \Omega \times I \setminus E$. Actually, by Vitali,

$$F_n \to \frac{\partial f}{\partial t}(\cdot, t) \quad \text{locally uniformly on } \Omega \quad (t \in I \setminus E \text{ fixed}).$$

So $\frac{\partial f}{\partial t}(\cdot, t)$ is holomorphic on $\Omega$ (Weierstrass). $\frac{\partial f}{\partial t}$ is measurable as a pointwise limit of continuous functions, and the boundedness property follows from the uniform Lipschitz property of $f$.

iii) Let $(z,t) \in \Omega \times I \setminus E$ be arbitrary, $(z_n, t_n) \in \Omega \times I \to (z,t)$ as $n \to \infty$. We have

$$\frac{f(\cdot, t_n) - f(\cdot, t)}{t_n - t} \to \frac{\partial f}{\partial t}(\cdot, t),$$

locally uniformly on $\Omega$, and so

$$\frac{f(z_n, t_n) - f(z_n, t)}{t_n - t} - \frac{\partial f}{\partial t}(z_n, t) = o(1), \qquad (t_n - t \neq 0).$$

So

$$\begin{aligned} f(z_n, t_n) - f(z,t) &= f(z_n, t_n) - f(z_n, t) + f(z_n, t) - f(z,t) \\ &= \frac{\partial f}{\partial t}(z_n, t)(t_n - t) + o(|t_n - t|) + \frac{\partial f}{\partial z}(z,t)(z_n - z) + o(|z_n - z|) \\ &= \frac{\partial f}{\partial t}(z,t)(t_n - t) + \frac{\partial f}{\partial z}(z_n - z) + o(|t_n - t| + |z_n - z|). \end{aligned}$$

iv) For any $n \in \mathbb{N}$, $\frac{\partial^n f}{\partial z^n}(\cdot, t)$ is holomorphic on $\Omega$ for $t \in I$. Suppose $\overline{B}(a,R) \subseteq \Omega$, $\gamma(t) = a + Re^{it}$. Then

$$\frac{\partial^n f}{\partial z^n}(z,t) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta, t)}{(\zeta - z)^{n+1}} d\zeta$$

for $z \in B(a,R)$, $t \in I$. By the Residue Theorem, if $z \in \overline{B}(a, R/2)$, $s, t \in J \subseteq I$ compact, then by the uniform Lipschitz property of $f$,

$$\left| \frac{\partial^n f}{\partial z^n}(z,s) - \frac{\partial^n f}{\partial z^n}(z,t) \right| \leq \frac{n!}{2\pi} \cdot 2\pi R \sup_{\zeta \in \partial B(a,R)} |f(\zeta, s) - f(\zeta, t)| \cdot \frac{1}{(R/2)^{n+1}} \leq C|s - t|,$$

so $t \to \dfrac{\partial^n f}{\partial z^n}(z,t)$ is uniform Lipschitz on $\overline{B}(a,R/2) \times J$. The uniform Lipschitz property of $\dfrac{\partial^n f}{\partial z^n}(z,t)$ follows from a covering argument.

Let $t \in I \setminus E$, $\{\delta_k\}$ be a sequence in $\mathbb{R}$ with $\delta_k \neq 0$, $\delta_k \to 0$. Then

$$\frac{f(\cdot,t+\delta_k) - f(\cdot,t)}{\delta_k} \to \frac{\partial f}{\partial t}(\cdot,t)$$

locally uniformly on $\Omega$; hence for $z \in B(a,R)$.

$$\begin{aligned}\frac{1}{\delta_k}\left[\frac{\partial^n f}{\partial z^n}(z,t+\delta_k) - \frac{\partial^n f}{\partial z^n}(z,t)\right] &= \frac{n!}{2\pi i}\int_\gamma \frac{f(\zeta,t+\delta_k)-f(\zeta,t)}{\delta_k}\frac{d\zeta}{(\zeta-z)^{n+1}} \\ &\to \frac{n!}{2\pi i}\int_\gamma \frac{\partial f(\zeta,t)}{\partial t}\frac{d\zeta}{(\zeta-z)^{n+1}} = \frac{\partial^n}{\partial z^n}\Big(\frac{\partial f}{\partial t}\Big)(z,t).\end{aligned}$$

This shows that $\dfrac{\partial}{\partial t}\Big(\dfrac{\partial^n f}{\partial z^n}\Big)(z,t)$ exists, and (11) holds.

v)
$$a_n(t) = \frac{1}{n!}\frac{\partial^n f}{\partial z^n}(0,t) \qquad \text{for } t \in I;$$

so $a_n$ is uniform Lipschitz on compact $J \subseteq I$ for each $n \in \mathbb{N}$ by iv). Moreover,

$$\dot{a}_n(t) = \frac{1}{n!}\frac{\partial}{\partial t}\Big(\frac{\partial^n f}{\partial z^n}\Big)(0,t) = \frac{1}{n!}\frac{\partial^n}{\partial z^n}\Big(\frac{\partial f}{\partial t}\Big)(0,t) \qquad \text{for } t \in I \setminus E.$$

So for $t \in I \setminus E$, the $n$-th Taylor coefficient of the holomorphic function of $z$, $\dfrac{\partial f}{\partial t}(\cdot,t)$ is given by $\dot{a}_n(t)$. (12) follows. □

**Theorem 4.13. (Main Theorem of Loewner Theory)** *Let $\{f_t\}_{t\in I}$, $I = [a,\infty)$ be a normalized Loewner chain, $\varphi_{s,t} = f_t^{-1} \circ f_s$, $f(z,t) := f_t(z)$. Then there exists $E \subseteq I$, $|E| = 0$, such that*

a) $V(z,t) := \lim\limits_{\varepsilon\to 0^+} \dfrac{\varphi_{t,t+\varepsilon}(z) - z}{\varepsilon}$ *exists for all $z \in \mathbb{D}$, $t \in I \setminus E$.*

b) $\dfrac{\partial f}{\partial t}(z,t)$ *exists for all $z \in \mathbb{D}$, $t \in I \setminus E$, and*

$$\frac{\partial f}{\partial t}(z,t) = -V(z,t)\frac{\partial f}{\partial z}(z,t). \qquad \textit{(Loewner-Kufarev equation)}$$

*Moreover, $V(z,t)$ has the following properties:*

i) *$V(\cdot,t)$ is holomorphic on $\mathbb{D}$ for each $t \in I \setminus E$,*

ii) *$V$ is measurable on $\Omega \times I$, and has the uniform bounded property: whenever $K \subseteq \mathbb{D}$, $J \subseteq I$ are compact, then there exists $M \geq 0$ such that $|V(z,t)| \leq M$ for $(z,t) \in K \times J \setminus E$.*

iii) *$V$ can be written in the form*

$$V(z,t) = -zp(z,t),$$

*where $p(\cdot,t) \in \mathcal{P}$ for $t \in I \setminus E$, i.e., $p(\cdot,t)$ is holomorphic in $\mathbb{D}$, $\operatorname{Re} p(\cdot,t) \geq 0$ and $p(0,t) = 1$.*

*Proof.* Since $f \in HL(\mathbb{D} \times I)$, there exists $E \subseteq I$, $|E| = 0$, such that $\dfrac{\partial f}{\partial t}(z,t)$ exists for $(z,t) \in \mathbb{D} \times I \setminus E$. Pick $(z,t) \in \mathbb{D} \times I \setminus E$ and $\varepsilon > 0$. Then $f_{t+\varepsilon}(\varphi_{t,t+\varepsilon}(z)) = f_t(z)$. Equivalently, $f(\varphi_{t,t+\varepsilon}(z), t+\varepsilon) = f(z,t)$. Differentiating with respect to $\varepsilon > 0$ and setting $\varepsilon = 0$, we obtain by the chain rule

$$0 = \frac{d}{d\varepsilon} f(\varphi_{t,t+\varepsilon}(z), t+\varepsilon)\bigg|_{\varepsilon=0} = \frac{\partial f}{\partial z}(z,t) \cdot \frac{\varphi_{t,t+\varepsilon}}{\partial \varepsilon}(z)\bigg|_{\varepsilon=0} + \frac{\partial f}{\partial t}(z,t).$$

Actually, this is true for any sublimit of

$$\frac{\partial \varphi_{t,t+\varepsilon}(z)}{\partial \varepsilon}\bigg|_{\varepsilon=0} = \lim_{\varepsilon \to 0} \frac{\varphi_{t,t+\varepsilon}(z) - z}{\varepsilon}.$$

Since $\dfrac{\partial f}{\partial z}(z,t) \neq 0$ ($f_t$ is conformal!), such a sublimt is unique. Since $\varepsilon \mapsto \varphi_{t,t+\varepsilon}$ is Lipschitz, the existence of

$$V(z,t) = \lim_{\varepsilon \to 0} \frac{\varphi_{t,t+\varepsilon}(z) - z}{\varepsilon}, \qquad (z \in \mathbb{D}, t \in I \setminus E),$$

follows, and

$$\frac{\partial f}{\partial z}(z,t) V(z,t) + \frac{\partial f}{\partial t}(z,t) = 0,$$

which is equivalent to the Loewner-Kufarev equation.

by Vitali,

$$\frac{\varphi_{t,t+\varepsilon_n}(z) - z}{\varepsilon_n} \to V(z,t)$$

locally uniformly for $z \in \mathbb{D}$, whenever $t \in I \setminus E$ fixed. So $V(\cdot,t)$ is holomorphic on $\mathbb{D}$; $V$ is measurable (pointwise limit of continuous functions), and has the uniform bounded property as follows form the uniform Lipschitz property of $(z,t) \mapsto \varphi_{s,t}(z)$.

$f(z,t)$ has the Taylor expansion

$$f(z,t) = a_0(t) + a_1(t)z + a_2(t)z^2 + \cdots, \qquad a_0(0t) \equiv w_0, \quad a_1(t) = e^t.$$

Let for fixed $t \in I \setminus E$, $V(z,t)$ has the Taylor expansion

$$V(z,t) = c_0(t) + c_1(t)z + c_2(t)z^2 + \cdots.$$

Then

$$\frac{\partial f}{\partial z}(z,t) = a_1(t) + 2a_2(t)z + \cdots,$$

and by Proposition 4.12 iv),

$$\frac{\partial f}{\partial t}(z,t) = \dot{a}_1(t)z + \dot{a}_2(t)z^2 + \cdots.$$

So

$$\dot{a}_1 z + \dot{a}_2 z^2 + \cdots = -(c_0 + c_1 z + \cdots)(a_1 + 2a_2 z + \cdots).$$

So $0 = -c_0 a_1 = -c_0 e^t$ equivalent to $c_0 = 0$, $\dot{a}_1 = -c_1 a_1$ equivalent to $e^t = c_1(t) \cdot e^t$ equivalent to $c_1(t) = -1$, i.e.,

$$V(z,t) = -zp(z,t),$$

where $p(\cdot,t)$ holomorphic and $p(0,t)=1$. By Schwarz's Lemma, $|\varphi_{t,t+\varepsilon}(z)| \le |z|$; so for $z \neq 0$,

$$\mathrm{Re}\Big(\frac{\varphi_{t,t+\varepsilon}(z)-z}{z}\Big) \le 0,$$

and for $z \neq 0$,

$$\mathrm{Re}\, p(z,t) = -\,\mathrm{Re}\Big(\frac{V(z,t)}{z}\Big) = -\lim_{\varepsilon\to 0^+} \mathrm{Re}\Big(\frac{\varphi_{t,t+\varepsilon}(z)-z}{\varepsilon z}\Big) \ge 0.$$

This inequality is also true for $z=0$ since $\mathrm{Re}\, p(0,t)=1$. □

**Corollary 4.14.** *Let $\{f_t\}$ be a normalized Loewner chain on $I=[a,\infty)$, $\varphi_{s,t}=f_t^{-1}\circ f_s$, $E\subseteq I$, $|E|=0$, $V(z,t)$ as in Theorem 4.13. Then*

i) $V(z,t) := \lim_{\varepsilon\to 0^+} \dfrac{\varphi_{t,t+\varepsilon}(z)-z}{\varepsilon} = \lim_{\varepsilon\to 0^+} \dfrac{\varphi_{t-\varepsilon,t}(z)-z}{\varepsilon}$ *for $z\in\mathbb{D}$, $t\in I\setminus E$.*

ii) $\dfrac{\partial\varphi_{s,t}(z)}{\partial t} = V(\varphi_{s,t}(z),t)$ *for $z\in\mathbb{D}$, $t\in[s,\infty)\setminus E$, $s\in I$ (left-hand derivative for $t=s$).*

iii) $\dfrac{\partial\varphi_{s,t}(z)}{\partial s} = -\varphi'_{s,t}(z)V(z,s)$ *for $z\in\mathbb{D}$, $s\in[0,t]\setminus E$, $t\in I$ (right-hand derivative for $s=t$).*

*The existence of limits post of the statement!*

*Proof.* i) For $t\in I\setminus E$, $\varepsilon>0$, $f_t\circ\varphi_{t-\varepsilon,t}(z) = f_{t-\varepsilon}(z)$. Differentiating with respect to $\varepsilon$ and setting $\varepsilon=0$:

$$f_t'(z)\cdot \frac{d}{d\varepsilon}\varphi_{t-\varepsilon,t}(z)\Big|_{\varepsilon=0} = -\dot f_t(z) = V(z,t)\cdot f_t'(z).$$

Hence $\lim_{\varepsilon\to 0^+}\dfrac{\varphi_{t-\varepsilon,t}(z)-z}{\varepsilon}$ exists and is equal to $V(z,t)$.

ii) For $s\in I$, $t\in[s,\infty)\setminus E$, $f_t\circ\varphi_{s,t}=f_s$, i.e, $f(\varphi_{s,t}(z),t)=f(z,s)$. Differentiating with respect to $t$ gives

$$f_t'(\varphi_{s,t}(z))\cdot\frac{\partial\varphi_{s,t}}{\partial t}(z) + \dot f_t(\varphi_{s,t}(z)) = 0,$$

equivalent to

$$\frac{\partial\varphi_{s,t}(z)}{\partial t} = -\frac{\dot f_t\circ\varphi_{s,t}}{f_t'\circ\varphi_{s,t}} = V(\varphi_{s,t}(z),t).$$

iii) For $t\in I$, $s\in[a,t]\setminus E$, $f_t\circ\varphi_{s,t}=f_s$, i.e., $f(\varphi_{s,t}(z),t)=f(z,s)$. Differentiating with respect to $s$ gives

$$f_t'(\varphi_{s,t}(z))\cdot\frac{\partial\varphi_{s,t}}{\partial s}(z) = \dot f_s(z) = -V(z,s)\cdot f_s'(z) = -V(z,s)\varphi'_{s,t}(z)f_t'(\varphi_{s,t}(z)).$$

So

$$\frac{\varphi_{s,t}}{\partial s}(z) = -\varphi'_{s,t}(z)\cdot V(z,s).$$

In all cases, existence of limits follows from the uniqueness of sublimits. □

### 4.15. Geometric interpretation

Figure 16: Geometric interpretation

$$\dot{f}_t(z) = -V(z,t)f_t'(z) = zp(z,t)f_t'(z).$$

Since $\operatorname{Re} p(z,r) > 0$, $zp(z,t)$ is a vector which points out of the disk $\overline{B}(0,|z|)$. Hence, $\dot{f}_t(z) = zp(z,t)f_t'(z)$ is a vector which points out of $f_t(\overline{B}(0,|z|))$

$$\frac{\partial \varphi_{s,t}}{\partial t}(z) = V(\varphi_{s,t}(z),t).$$

So $t \mapsto \varphi_{s,t}(z)$ is an integral curve of the vector field $V(z,t)$. $z \mapsto \varphi_{s,t}(z)$ is a map which shrinks $\mathbb{D}$ for large $t$, with $\varphi_{s,t}'(0) = e^{s-t}$.

Figure 17: Shrink

$$\frac{\partial \varphi_{s,t}}{\partial s}(z) = -\varphi_{s,t}(z)V(z,s).$$

So

$$\varphi_{s-\varepsilon,s}(z) \simeq z + \varepsilon V(z,s).$$

We have

$$\varphi_{s-\varepsilon,t}(z) \simeq \varphi_{s,t}(z) + \varepsilon\varphi_{s,t}'(z)V(z,s).$$

Figure 18: transfer

# 5 Existence results for Loewner chains and applications

**Proposition 5.1.** *Let $\{f_t^n\}$ be a sequence of normalized Loewner chains on $I = [a,\infty)$, $f_t^n(0) = w_0 \in \mathbb{C}$, $(f_t^n)'(0) = e^t$, $t \in I$. Then $\{f_t^n\}$ subconverges to a Loewner Chain as $n \to \infty$; more precisely, there exists a sequence $\{n_k\}$ with $n_k \to \infty$ as $k \to \infty$ and a normalized Loewner chain $\{f_t\}_{t\in I}$ such that $f_t^{n_k} \to f_t$ locally uniformly on $\mathbb{D}$ as $k \to \infty$, for all $t \in I$.*

*Proof.* Wlog, $w_0 = 0$. Let $z_l = 1/l$, $l \geq 2$. Then $z_l \to 0 \in \mathbb{D}$ as $l \to \infty$. For fixed $l \in \mathbb{N}$, the maps $t \in [0,\infty) \mapsto f_t^n(z)$, $n \in \mathbb{N}$, are uniform Lipschitz (cf. Lemma 4.10) and uniformly bounded (Koebe) on compact set $J \subseteq I$. In particular, the family $\{t \mapsto f_t^n(z_l)\}_{n\in\mathbb{N}}$ is equicontinous and uniformly bounded at each $t_0 \in I$. Hence, by the Arzela-Ascoli Theorem, there exists a subsequence that converges locally uniformly on $I$ and in particular pointwisely on $I$.

Applying this successively for each $l = 2, 3, \ldots$, and passing to a diagonal subsequence, we find a sequence $\{n_k\}$ in $\mathbb{N}$ with $n_k \to \infty$ as $k \to \infty$ such that $\{f_t^{n_k}(z_l)\}$ converges as $k \to \infty$ for all $t \in I$, $l \geq 2$.

Fix $t \in I$. Then $e^{-t}f_t^{n_k} \in \mathcal{S}$, and so these functions form a normal family. Since we have pointwise convergence at each $z_l \in \mathbb{D}$, $l \geq 2$, by Vitali's Theorem, $\{f_t^{n_k}\}$ converges locally uniformly on $\mathbb{D}$ to some limit function $f_t \in H(\mathbb{D})$. So $f_t^{n_k} \to f_t$ locally uniformly on $\mathbb{D}$ as $k \to \infty$ for each $t \in I$.

It is suffices to show that $\{f_t\}_{t\in I}$ is a normalized Loewner chain.

$$f_t(0) = \lim_{k\to\infty} f_t^{n_k}(0) = 0$$

and

$$f_t'(0) = \lim_{k\to\infty} (f_t^{n_k})'(0) = e^t \neq 0 \qquad \text{for } t \in I. \tag{14}$$

By Hurwitz, $f_t$ is a conformal map $f_t : \mathbb{D} \leftrightarrow \Omega_t = f_t(\mathbb{D})$. If $s, t \in I$ and $s \leq t$, then

$$\Omega_s^{n_k} := f_s^{n_k}(\mathbb{D}) \to \Omega_s, \quad \Omega_t^{n_k} := f_t^{n_k}(\mathbb{D}) \to \Omega_t, \quad \text{w.r.t. } w_0,$$

and $\Omega_s^{n_k} \subseteq \Omega_t^{n_k}$. So

$$\Omega_s \subseteq \Omega_t. \tag{15}$$

A combination of (14) and (15) implies the Lipschitz estimates for $t \mapsto f_t(z)$ as in Lemma 4.10 ii) ($\varphi_{s,t} = f_t^{-1} \circ f_s$ is defined, etc.). Hence $f_{t_n} \to f_t$ locally uniformly on $\mathbb{D}$ whenever $t_n \in I \to t \in I$. So $\{f_t\}$ is a Loewner chain. □

**Corollary 5.2.** *Let $f \in \mathcal{S}$. Then there exists a Loewner chain $\{f_t\}_{t\in[0,\infty)}$ with $w_0 = 0$ such that $f_0 = f$.*

*Proof.* For $n \in \mathbb{N}$, $n \geq 2$, let $r_n = (1 - 1/n) \in (0, 1)$, and

$$f^n(z) = \frac{1}{r_n} f(r_n z), \qquad z \in \mathbb{D}.$$

Then $f^n(0) = 0$, $(f^n)'(0) = 1$, and so $f^n \in \mathcal{S}$. $f^n$ is a conformal map from $\mathbb{D}$ onto the Jordan region $\Omega^n = f^n(\mathbb{D}) = f(B(0, r_n))$. So $\Omega^n$ can be embedded in a Loewner chain; equivalently, there exists a normalized Loewner chain $\{f_t^n\}_{t\in[0,\infty)}$ with $f_t^n(0) = 0$, $(f_t^n)'(0) = e^t$ for $t \in I = [0, \infty)$, and $f_0^n = f^n$. By Proposition 5.1, the sequence $\{f_t^n\}$ of Loewner chains subconverges to a normalized Loewner chain $\{f_t\}$; i.e., for some sequence $\{n_k\}$ with $n_k \to \infty$, we have $f_t^{n_k} \to f_t$ locally uniformly on $\mathbb{D}$ for each $t \in I$. In particular, $f_0^{n_k} = f^{n_k} \to f_0$ locally uniformly on $\mathbb{D}$. On the other hand,

$$f^{n_k}(z) = \frac{1}{r_n} f(r_n z) \to f(z)$$

locally uniformly for $z \in \mathbb{D}$. So $f_0 = f$, the claim follows. □

### 5.3. Loewner chains and Taylor coefficients

Let $f \in \mathcal{S}$ be arbitrary. $f : \mathbb{D} \to \Omega = f(\mathbb{D})$ conformal, $f(0) = 0$, $f'(0) = 1$. By Corollary 5.2, there exists a normalized Loewner chian $\{f_t\}_{t\in[0,\infty)}\}$ such that $f_0 = f$, $f_t(0) = 0$, $f_t'(0) = e^t$.

$$f_t(z) = \sum_{n=1}^{\infty} a_n(t) z^n, \qquad t \in [0, \infty), \qquad \text{with } a_1(t) = e^t.$$

Let $f(z, t) = f_t(z)$, $I = [0, \infty)$. There exists $E \subseteq [0, \infty)$ with $|E| = 0$ such that

$$\frac{\partial f}{\partial t}(z, t) = zp(z, t)\frac{\partial f}{\partial z}(z, t), \qquad z \in \mathbb{D}, t \in I \setminus E,$$

where $f \in HL(\mathbb{D} \times I)$, $p(\cdot, t) \in \mathcal{P}$ for $t \in I \setminus E$, i.e., $p(\cdot, t) \in \mathrm{H}(\mathbb{D})$, $p(0, t) = 1$, and $\operatorname{Re} p(\cdot, t) \geq 0$,

$$p(z, t) = 1 + \sum_{n=1}^{\infty} c_n(t) z^n, \qquad z \in \mathbb{D}, t \in I \setminus E.$$

From Proposition 4.12,

$$\frac{\partial f}{\partial t}(z, t) = \sum_{n=1}^{\infty} \dot{a}_n(t) z^n, \qquad z \in \mathbb{D}, t \in I \setminus E.$$

Fix $t \in I \setminus E$. Then

$$\begin{aligned} \sum_{n=1}^{\infty} \dot{a}_n(t) z^n &= z \left( 1 + \sum_{n=1}^{\infty} c_n(t) z^n \right) \left( \sum_{n=1}^{\infty} n a_n(t) z^{n-1} \right) \\ &= \sum_{n=1}^{\infty} \left( n a_n(t) + \sum_{k=1}^{n-1} k a_k(t) c_{n-k}(t) \right) z^n. \end{aligned}$$

Comparing coefficients, we get

$$\dot{a}_n(t) = n a_n(t) + \sum_{k=1}^{n-1} k a_k(t) c_{n-k}(t), \qquad t \in I \setminus E, n \in \mathbb{N}.$$

Each $a_n$ is locally Lipschitz (cf. Proposition 4.12), $c_n$ is measurable (homework!). Moreover, $|c_n(t)| \leq 2$ for $n \in \mathbb{N}$, $t \in I \setminus E$ (Theorem 4.9 (iv)). Noting that $h_t := e^{-t} f_t \in \mathcal{S}$ and $\mathcal{S}$ is a normal family, there exists $C_n \geq 0$ such that

$$\left| e^{-t} a_n(t) \right| = \left| \frac{h_t^{(n)}(0)}{n!} \right| \leq C_n, \quad \text{for } t \in I,$$

hence $e^{-nt} a_n(t) \to 0$ as $t \to \infty$ for $n \geq 2$.

$$\frac{d}{dt} \left( e^{-nt} a_n(t) \right) = e^{-nt} \dot{a}_n(t) - e^{-nt} n a_n(t) = \sum_{k=1}^{n-1} e^{-nt} k a_k(t) c_{n-k}(t), \quad \text{for } t \in I \setminus E.$$

For $s \geq 0$, $n \geq 2$,

$$-e^{-ns} a_n(s) = \lim_{u \to \infty} \int_s^u \frac{d}{dt} \left( e^{-nt} a_n(t) \right) dt = \sum_{k=1}^{\infty} k \int_s^{\infty} e^{-nt} a_k(t) c_{n-k}(t) dt.$$

So

$$a_n(s) = -e^{ns} \sum_{k=1}^{n-1} k \int_s^{\infty} e^{-nt} a_k(t) c_{n-k}(t) dt, \qquad s \geq 0, n \geq 2.$$

Taking $s = 0$, $n = 2$,

$$a_2 = a_2(0) = -\int_0^{\infty} e^{-2t} a_1(t) c_1(t) dt = -\int_0^{\infty} e^{-t} c_1(t) dt.$$

Taking $s = 0$, $n = 3$,

$$\begin{aligned}
a_3 = a_3(0) &= -\sum_{k=1}^{2} k \int_0^\infty e^{-3t} a_k(t) c_{3-k}(t) dt \\
&= -\int_0^\infty e^{-2t} c_2(t) dt - 2\int_0^\infty e^{-3t} a_2(t) c_1(t) dt \\
&= -\int_0^\infty e^{-2t} c_2(t) dt + 2\int_0^\infty e^{-3t} e^{2t} \left( \int_t^\infty e^{-u} c_1(u) du \right) c_1(t) dt \\
&= -\int_0^\infty e^{-2t} c_2(t) dt + 2\int_0^\infty e^{-t} \left( \int_t^\infty e^{-u} c_1(u) du \right) c_1(t) dt \\
&= -\int_0^\infty e^{-2t} c_2(t) dt + \int_0^\infty \int_0^\infty e^{-t} c_1(t) e^{-u} e_1(u) dt du \\
&= -\int_0^\infty e^{-2t} c_2(t) dt + \left( \int_0^\infty e^{-t} c_1(t) dt \right)^2 .
\end{aligned}$$

**Corollary 5.4.** *Let $f \in \mathcal{S}$, $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$. Then $|a_2| \le 2$, $|a_3| \le 3$.*

*Proof.* Using notations from 5.3, we have

$$a_2 = -\int_0^\infty e^{-t} e_1(t) dt.$$

Now, $|c_1(t)| \le 2$ (cf. Theorem 4.9 iv)), and

$$|a_2| \le \int_0^\infty e^{-t} |c_1(t)| dt \le 2\int_0^\infty e^{-t} dt = 2.$$

(case of equality can be analyzed!)

By rotation invariance ($f \in \mathcal{S} \longleftrightarrow e^{i\theta} f(ze^{-i\theta}) \in \mathcal{S}$), wlog, we assume $a_3 \ge 0$. Then, using Theorem 4.9 iv),

$$\begin{aligned}
a_3 = \operatorname{Re} a_3 &\le -\int_0^\infty e^{-2t} \operatorname{Re} c_2(t) dt + \left( \int_0^\infty e^{-t} \operatorname{Re} c_1(t) dt \right)^2 \\
&\le -\int_0^\infty e^{-2t} \operatorname{Re} c_2(t) dt + \int_0^\infty e^{-t} (\operatorname{Re} c_1(t))^2 dt \qquad \text{(Cauchy-Schwarz)} \\
&\le 2\int_0^\infty e^{-t} dt + \int_0^\infty (\operatorname{Re} c_2(t))(e^{-t} - e^{-2t}) dt \qquad ((\operatorname{Re} c_1)^2 \le 2 + \operatorname{Re} c_2) \\
&\le 2 + 2\int_0^\infty (e^{-t} - e^{-2t}) dt \qquad (|c_2| \le 2 \text{ and } e^{-t} - e^{-2t} \ge 0) \\
&= 2 + 2 + 2\left[ \frac{1}{2} e^{-2t} \right]_0^\infty = 3.
\end{aligned}$$

□

**Lemma 5.5.** *Let $p \in \mathcal{P}$. Then*

(i) $|p'(z)| \le \dfrac{2}{(1-|z|)^2}, \quad z \in \mathbb{D}$,

(ii) $|p(u) - p(v)| \le \dfrac{2|u-v|}{(1-r)^2}. \quad u, v \in \overline{B}(0,r),\ r \in (0,1)$.

*Proof.* Let $z_0 \in \mathbb{D}$, $r \in (0,1)$ and $r > |z_0|$, $\gamma(t) = re^{it}$, $t \in [0, 2\pi]$. Then

$$p'(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{p(\zeta)}{(\zeta - z_0)^2} d\zeta.$$

On the other hand, there exists a probability measure $\mu$ on $\partial\mathbb{D}$ such that

$$p(\zeta) = \int_{\partial\mathbb{D}} \frac{\eta + \zeta}{\eta - \zeta} d\mu(\eta), \qquad \text{for } \zeta \in \mathbb{D}.$$

Let $K_\eta(\zeta)$ denote $(\eta + \zeta)/(\eta - \zeta)$. By Fubini,

$$\begin{aligned} p'(z_0) &= \frac{1}{2\pi i} \int_\gamma \frac{p(\zeta)}{(\zeta - z_0)^2} d\zeta = \frac{1}{2\pi i} \int_\gamma \int_{\partial\mathbb{D}} \frac{K_\eta(\zeta)}{(\zeta - z_0)^2} d\mu(\eta) d\zeta \\ &= \int_{\partial\mathbb{D}} \left[ \frac{1}{2\pi i} \int_\gamma \frac{K_\eta(\zeta)}{(\zeta - z_0)^2} d\zeta \right] d\mu(\eta) = \int_{\partial\mathbb{D}} K'_\eta(z_0) d\mu(\eta), \end{aligned}$$

(so we can differentiate under the integral sign in the Herglotz formula)

here

$$K'_\eta(z_0) = \frac{d}{dz} \Big( \frac{\eta + z}{\eta - z} \Big) \Big|_{z = z_0} = \frac{2}{(\eta - z_0)^2}, \quad |K'_\eta(z_0)| \leq \frac{2}{(1 - |z_0|)^2},$$

and

$$|p'(z_0)| \leq \int_{\partial\mathbb{D}} \frac{2}{(1 - |z_0|)^2} d\mu(\eta) = \frac{2}{(1 - |z_0|)^2}.$$

(ii) follows from (i). □

**Lemma 5.6.** *Let $I = [a, \infty)$, $p : \mathbb{D} \times I \to \mathbb{C}$ be a.e. defined, $p$ be measureable, $p(\cdot, t) \in \mathcal{P}$ for a.e.$t \in I$. Let $J = [a, b] \subseteq I$, and suppose $u, v : J \to \mathbb{D}$ are absolute continuous and solutions of the ODE*

$$\dot{w}(t) = -w(t) p(w(t), t) \qquad \textit{for a.e. } t. \tag{16}$$

*If $u(t_0) = v(t_0)$ for some $t_0 \in J$, then $u = v$.*

*Proof.* 1) For a solution $w : J \to \mathbb{D}$, $t \mapsto |w(t)|$ is decreasing:

$$\begin{aligned} \frac{d}{dt} |w(t)|^2 = \frac{d}{dt} w(t) \overline{w(t)} &= \dot{w}(t) \overline{w(t)} + w(t) \overline{\dot{w}(t)} \\ &= -|w(t)|^2 p(w(t), t) - |w(t)|^2 \overline{p(w(t), t)} \\ &= -|w(t)|^2 \operatorname{Re} p(w(t), t) \leq 0 \qquad \text{for a.e. } t. \end{aligned}$$

So $|u(t)|, |v(t)| \leq r := \max\{|u(a)|, |v(a)|\} < 1$.

2)
$$\begin{aligned} &|u(t) p(u(t), t) - v(t) p(v(t), t)| \\ &\leq |u(t)| |p((u(t), t) - p(v(t), t)| + |u(t) - v(t)| |p(v(t), t)| \\ &\leq 1 \cdot \frac{2}{(1 - r)^2} |u(t) - v(t)| + \frac{2}{1 - r} |u(t) - v(t)| \\ &\leq K |u(t) - v(t)|, \end{aligned}$$

for a.e. $t$, where $K$ is independent of $t$. Let $D(t) := (u(t) - v(t))^2$, $t \in J$. Then $D$ is absolute continuous, and

$$\begin{aligned}\Big|\frac{d}{dt}D(t)\Big| &\leq 2|\dot{u}(t) - \dot{v}(t)||u(t) - v(t)| \\ &= 2|u(t)p(u(t),t) - v(t)p(v(t),t)||u(t) - v(t)| \\ &\leq 2K|u(t) - v(t)|^2 = K'D(t).\end{aligned}$$

Hence

$$D(t) \leq e^{K'|t-t_0|}D(t_0) \quad \text{for } t \in J. \qquad \text{(special case of Gronwell's inquality)}$$

Since $D(t_0) = 0$, we conclude $D(t) \equiv 0$ and so $u \equiv v$. □

**Theorem 5.7.** *Let $I = [a,b) \subseteq \mathbb{R}$, $V : \mathbb{D} \times I \to \mathbb{C}$ be a.e. defined measurable function such that*
a) *$V(z,\cdot)$ is a.e defined and measurable for each $z \in \mathbb{D}$,*
b) *$V(\cdot,t)$ is holomorphic on $\mathbb{D}$ for a.e. $t \in I$ and*

$$V(z,t) = -zp(z,t) \qquad \text{for } z \in \mathbb{D},$$

*where $p(\cdot,t) \in \mathcal{P}$. Then for each $z \in \mathbb{D}$, $s \in I$, there exists a unique map $w : [s,\infty) \to \mathbb{D}$ such that*
i) *$w$ is Lipschitz on $[s,\infty)$,*
ii) *$w(s) = z$ (initial condition),*
iii) *$\dot{w}(t) = V(w(t),t)$ for a.e. $t \in I$.*

*Proof.* Need a technical lemma that will be formulated afterward!
Idea of proof: Picard-Lindelöf iteration scheme!
Le $z \in \mathbb{D}$, $s \in I$ fixed. Define $w_0(t) \equiv 0$ and

$$w_{n+1}(t) = z \cdot \exp\Big(-\int_s^t p(w_n(u),u)du\Big), \qquad \text{for } n \in \mathbb{N}_0, t \geq s.$$

(so $w_1(t) = ze^{s-t}$.)
i) $|w_n(t)| \leq r := |z|$, $t \geq s$, $n \in \mathbb{N}$ (note $\operatorname{Re} p \geq 0$).
ii) $w_n$ is $L$-Lipschitz on $[s,\infty)$ with $L = 2r/(1-r)$:

$$\begin{aligned}|w_{n+1}(t_2) - w_{n+1}(t_1)| &= |z|\Big|\exp\Big(-\int_s^{t_2}\cdots\Big) - \exp\Big(-\int_s^{t_1}\cdots\Big)\Big| \\ &\leq |z|\Big|\int_s^{t_2}\cdots - \int_s^{t_1}\cdots\Big| = |z|\Big|\int_{t_1}^{t_2} p(w_n(u),u)du\Big| \\ &\leq \frac{2r}{1-r}|t_2 - t_1|, \qquad t_1 \geq t_1 \geq s,\end{aligned}$$

here we have used the fact $|e^{-a} - e^{-b}| \leq |a - b|$ for $\operatorname{Re} a, \operatorname{Re} b \geq 0$, and

$$p(w_n(u),u) \leq \frac{2}{1 - |w_n(u)|} \leq \frac{2}{1-r}.$$

iii) $|w_{n+1}(t) - w_n(t)| \leq \dfrac{2^n(t-s)^n}{(1-r)^{2n}n!}$, for $n \in \mathbb{N}_0, t \geq s$:

By induction: for $n = 0$,

$$|w_1(t) - w_0(t)| = e^{s-t}|z| \leq 1, \qquad \text{OK.}$$

$n \to n+1$,

$$\begin{aligned}
|w_{n+1}(t) - w_n(t)| &= |z| \Big| \exp\Big(-\int_s^t p(w_n(u),u)du\Big) - \exp\Big(-\int_s^t p(w_{n-1}(u),u)du\Big) \Big| \\
&\leq |z| \Big| \int_s^t |p(w_n(u),u) - p(w_{n-1}(u),u)| du \Big| \\
&\leq |z| \frac{2}{(1-r)^2} \int_s^t |w_n(u) - w_{n-1}(u)| du \\
&\leq |z| \frac{2}{(1-r)^2} \int_s^t \frac{2^n (u-s)^n}{(1-r)^{2n} n!} du = \frac{2^{n+1}}{(1-r)^{2(n+1)}} \cdot \frac{(t-s)^{n+1}}{(n+1)!}.
\end{aligned}$$

So

$$w(t) := \lim_{n\to\infty} w_n(t) = w_0(t) + \sum_{n=1}^{\infty} (w_n(t) - w_{n+1}(t))$$

exists for each $t \in I$, convergence uniformly on compact subsets $J \subseteq I$, i.e., $w_n \to w$ locally uniformly on $I$.

Thus, $w$ is $L$-Lipschitz on $I$, $|w(t)| \leq r < 1$ for $t \in I$, $p(w_n(u),u) \to p(w(u),u)$ for a.e. $u \in I$. Since $|p(w_n(u),u)| \leq 2/(1-r)$, so

$$\int_s^t p(w_n(u),u)du \to \int_s^t p(w(u),u)du \qquad \text{for each } t \in [s,\infty)$$

by the Lebesgue dominated convergence theorem.

For each $t \in I$,

$$\begin{aligned}
w(t) = \lim_{n\to\infty} w_{n+1}(t) &= \lim_{n\to\infty} z \exp\Big(-\int_s^t p(w_n(u),u)du\Big) \\
&= z \exp\Big(-\int_s^t p(w(u),u)du\Big) \qquad \text{for } t \in [s,\infty).
\end{aligned}$$

So $w(s) = z$, $\dot{w}(t)$ exists for a.e. $t \in [s,\infty)$, and

$$\dot{w}(t) = -z \exp\Big(-\int_s^t p(w(u),u)du\Big) \cdot p(w(t),t) = -w(t)p(w(t),t) = V(w(t),t).$$

Existence of $w$ follows.

Uniqueness is clear by 5.6 □

**Corollary 5.8.** *For fixed $z \in \mathbb{D}$, $s,t \in I$ with $s \leq t$, let $\varphi_{s,t}(z) = w(t)$, where $w$ is as in Theorem 5.7. Then*

i) *$\varphi_{s,t}(\cdot)$ is holomorphic and injective on $\mathbb{D}$, $\varphi_{s,t}(\mathbb{D}) \subseteq \mathbb{D}$.*
ii) *$\varphi_{s,t}(0) = 0$, $\varphi'_{s,t}(0) = e^{s-t}$.*
iii) *$\varphi_{s,u} = \varphi_{t,u} \circ \varphi_{s,t}$, $\quad 0 \leq s \leq t \leq u < \infty$.*

iv) $f_s(z) := \lim_{t\to\infty} e^t \varphi_{s,t}(z)$ *exists for* $z \in \mathbb{D}$, $s \in I$. *Moreover,* $e^t\varphi_{s,t} \to f_s$ *locally uniformly on* $\mathbb{D}$.

v) $\{f_s\}_{s\in I}$ *is a Loewner chain with*

$$\dot{f}_s(z) = V(z, s) f_s'(z) \qquad \text{for } z \in \mathbb{D} \text{ and a.e. } s \in I.$$

*Proof.* As in the proof of Theorem 5.7, define for $z \in \mathbb{D}$, $t \geq s$,

$$\begin{aligned} &w_0(z,t) \equiv 0, \\ &w_{n+1}(z,t) := z \exp\Big(-\int_s^t p(w_n(z,u), u)du\Big). \end{aligned}$$

Using induction and Morera, one shows that $w_n(z,t)$ is holomorphic on $\mathbb{D}$ for each $t \in [s,\infty)$.

Since $|w_n(z,t)| \leq |z|$, we have $\{w_n(\cdot,t)\}_{n\in\mathbb{N}}$ is a normal family for each $t \in [s,\infty)$. Then $w_n(z,t) \to w(z,t) = \varphi_{s,t}(z)$ pointwise on $\mathbb{D}$ for each $t \in [s,\infty)$, convergence is locally uniformly on $\mathbb{D}$ by Vitali.

i) Hence $w(\cdot,t) = \varphi_{s,t}$ is holomorphic on $\mathbb{D}$ for $s,t \in I$, $s \leq t$. Let $s, t_0 \in I$, $s \leq t_0$, $z_1, z_2 \in \mathbb{D}$, and suppose $\varphi_{s,t_0}(z_1) = \varphi_{s,t_0}(z_2)$, equivalently $w(z_1,t_0) = w(z_2,t_0)$. Then by Lemma 5.6, $w(z_1,t) = w(z_2,t)$ for all $t \geq s$; hence $z_1 = w(z_1,s) = w(z_2,s) = z_2$. So $\varphi_{s,t_0}$ is injective on $\mathbb{D}$.

ii) $w(0,t) \equiv 0$ solves ODE; so $\varphi_{s,t}(0) = 0$.

$$\varphi_{s,t}(z) = z \exp\Big(-\int_s^t p(\varphi_{s,u}(z), u)du\Big).$$

So

$$\varphi_{s,t}'(0) = \exp\Big(-\int_s^t p(\varphi_{s,u}(0), u)du\Big) = \exp(-(t-s)) = e^{s-t}. \tag{17}$$

iii) Let $v(u) := \varphi_{s,u}(z)$, $\tilde{v}(u) := \varphi_{t,u}(\varphi_{s,t}(z))$, where $z \in \mathbb{D}$, $s \leq t \leq u$ fixed. Then $v(t) = \varphi_{s,t}(z)$, $\tilde{v}(t) = \varphi_{t,t}(\varphi_{s,t}(z)) = \varphi_{s,t}(z)$, since $\varphi_{t,t}(z) = z$. So $v, \tilde{v}$ have the same initial values at time $u = t$. They satisfy equations

$$\dot{v}(u) = V(v(u), u), \quad \dot{\tilde{v}}(u) = V(\tilde{v}(u), u) \qquad \text{for a.e. } u.$$

So $v(u) \equiv \tilde{v}(u)$ for $u \geq t$ by Lemma 5.6, i.e.,

$$\varphi_{s,u}(z) = \varphi_{t,u}(\varphi_{s,t}(z)) \qquad \text{for } z \in \mathbb{D}, s \leq t \leq u.$$

iv) By (17),

$$e^{t-s}\varphi_{s,t}(z) = z \exp\Big(\int_s^t [1 - p(\varphi_{s,u}(z), u)]du\Big) \in \mathcal{S},$$

so by Koebe,

$$|\varphi_{s,t}(z)| \leq \frac{e^{s-t}|z|}{(1-|z|)^2}, \qquad z \in \mathbb{D}.$$

So

$$\begin{aligned} |1 - p(\varphi_{s,u}(z), u)| &= |p(0,u) - p(\varphi_{s,u}(z), u)| \\ &\leq |\varphi_{s,u}(z)| \frac{2}{(1-|z|)^2} \qquad \text{(Lemma 5.5)} \\ &\leq e^{s-u} \frac{2|z|}{(1-|z|)^4} \leq Ce^{-u}, \qquad \text{for fixed } s, z. \end{aligned}$$

So

$$\int_s^\infty |1 - p(\varphi_{s,u}(z), u)| du < \infty$$

with uniform convergence in $z$ on compact subsets of $\mathbb{D}$. Hence

$$\begin{aligned} f_s(z) &:= \lim_{t\to\infty} e^t \varphi_{s,t}(z) = \lim_{t\to\infty} e^s e^{t-s} \varphi_{s,t}(z) \\ &= e^s \cdot z \exp\Big(\int_s^\infty [1 - p(\varphi_{s,u}(z), u)] du\Big) \end{aligned}$$

exists with locally uniform convergence in $z \in \mathbb{D}$. So $f_s \in H(\mathbb{D})$,

$$\begin{aligned} f_s(0) &= \lim_{t\to\infty} e^t \varphi_{s,t}(0) = 0, \\ f_s'(0) &= \lim_{t\to\infty} e^t \varphi_{s,t}'(0) = e^s. \end{aligned}$$

Since $e^t \varphi_{s,t}$ is injective on $\mathbb{D}$, $f_s$ is injective on $\mathbb{D}$ by Hurwitz.

For $z \in \mathbb{D}$, $s \leq t$,

$$f_t(\varphi_{s,t}(z)) = \lim_{u\to\infty} e^u \varphi_{s,u}(\varphi_{s,t}(z)) = \lim_{u\to\infty} e^u \varphi_{s,u}(z) = f_s(z).$$

So $f_t \circ \varphi_{s,t} = f_s$ for $s \leq t$. Hence $\Omega_t = f_t(\mathbb{D}) \supseteq f_t(\varphi_{s,t}(\mathbb{D})) = \Omega_s$. (Strict inclusion for $s < t$ comes from $\varphi_{s,t}'(0) = e^{s-t} < 1$ and $\varphi_{st}$ is a conformal map.) As in Proposition 5.1, we conclude that $\{f_s\}_{s\in I}$ is a Loewner chain.

Since $\{f_s\}_{s\in I}$ is a Loewner chain, $(z,t) \mapsto f(z,t) \in HL(\mathbb{D} \times I)$. Since $f(\varphi_{a,t}(z), t) = f_a(z)$, there exists $E \subseteq I - [a, \infty)$, $|E| = 0$, such that

$$\begin{aligned} 0 &= \frac{d}{dt} f_a(z) = \frac{d}{dt} f(\varphi_{a,t}(z), t) \\ &= f_t'(\varphi_{a,t}(z)) \cdot \frac{d}{dt}\varphi_{a,t}(z) + \dot{f}_t(\varphi_{a,t}(z)). \end{aligned}$$

Since $\frac{d}{dt}\varphi_{a,t}(z) = V(\varphi_{a,t}(z), t)$,

$$\dot{f}_t(w) = -V(w,t) \cdot f_t'(w), \qquad \text{for } t \in I \setminus E, w \in \varphi_{a,t}(\mathbb{D}) \subseteq \mathbb{D}.$$

We may assume that $\dot{f}_t(\cdot)$ and $V(\cdot, t)$ are holomorphic for $t \in I \setminus E$. Then by the uniqueness Theorem,

$$\dot{f}_t(z) = -V(z,t) \cdot f_t'(z), \qquad \text{for } z \in \mathbb{D}, t \in I \setminus E. \qquad \square$$

Continuity of $w_n(z,t)$ in $z$ for $t$ fixed:

$$\begin{aligned} &w_0(z,t) \equiv 0; \\ &w_{n+1}(z,t) = z \exp\Big(-\int_s^t p(w_n(z,u), u) du\Big). \end{aligned}$$

By induction on $n$. $n \to n+1$:

$z_k \in \mathbb{D} \to z_0 \in \mathbb{D}$, $|z_k| \leq r < 1$, $w_n(z_k, u) \to w_n(z_0, u)$ as $n \to \infty$ for each $u \in [s,t]$. Moreover, $|w_n(z_k, u)| \leq r$ and so

$$p(w_n(z_k, u), u) \leq \frac{1+r}{1-r}.$$

So

$$\int_s^t p(w_n(z_k,u),u)du \to \int_s^t p(w_n(z_0,u),u)du$$

by the Lebesgue dominated convergence theorem.

In the proof of Theorem 5.7, the following fact was used.

**Lemma 5.9.** *Let $U \subseteq \mathbb{R}^d$ be open, $M \subseteq \mathbb{R}^d$ be measurable, $g : U \times M \to \mathbb{C}$ be a.e. defined such that*

i) *$g(\cdot, t)$ is continuous on $U$ for a.e. $t \in M$,*

ii) *$g(z, \cdot)$ is a.e. defined on $M$ and measurable.*

*Let $\phi : M \to U$ be measurable. Then $h : M \to \mathbb{C}$ a.e. defined by $h(t) := g(\phi(t), t)$ for $t \in M$ is measurable.*

*Outline of Proof.* I. For each $n \in \mathbb{N}$, pick a countable open covers $\mathcal{U}_n = \{U_{n,k} : k \in \mathbb{N}\}$ of $U$ such that $U_{n,k} \Subset U$ and

$$\operatorname{mesh}(\mathcal{U}_n) = \sup\{\operatorname{diam}(U_{n,k}) : k \in \mathbb{N}\} \to 0 \qquad \text{as} \quad n \to \infty.$$

Pick $z_{n,k} \in U_{n,k}$ and let $\{\varphi_{n,k} : k \in \mathbb{N}\}$ be a partition of unity subordinate to $\mathcal{U}_n$. For $f \in C(U)$, define

$$T_n f := \sum_{k \in \mathbb{N}} f(z_{n,k})\varphi_{n,k} \in C(U).$$

Then $T_n f \to f$ locally uniformly on $U$ for all $f \in C(U)$.

For $z \in U$,

$$\begin{aligned} |h(z) - T_n h(z)| &\le \sum_{k \in \mathbb{N}} |h(z) - h(z_{n,k})|\varphi_{n,k}(z) \\ &\le \sup\{|h(u) - h(u')| : |u - u'| \le \operatorname{mesh}(\mathcal{U}_n)\}. \end{aligned}$$

II. There exists $E \subseteq M$, $|E| = 0$ such that $g(\cdot, t) \in C(U)$ for $t \in M \setminus E$. Then

$$T_n g(z,t) = \sum_{k \in \mathbb{N}} g(z_{n,k}, t)\varphi_{n,k}(z) \to g(z,t) \qquad \text{as} \quad n \to \infty$$

for $z \in U$, $t \in M \setminus E$. So for a.e. $t \in M$,

$$\sum_{k \in \mathbb{N}} g(z_{n,k}, t)\varphi_{n,k}(\psi(t)) \to g(\psi(t), t) = h(t) \qquad \text{as} \quad n \to \infty.$$

So $h$ is measurable. □

**Lemma 5.10.** *Let $f \in \mathcal{S}$. Then*

$$|f(z) - z| \le C\frac{|z|^2}{(1 - |z|)^2} \qquad \textit{for} \quad z \in \mathbb{D},$$

*where $C$ is an absolute constant independent of $f$.*

*Proof.* Define

$$g(z) = \frac{1}{z^2}(f(z) - z), \qquad z \in \mathbb{D}.$$

Then $g \in H(\mathbb{D})$ (0 is a removable singularity). Pick $0 < r < 1$. Then by Koebe and Maximum principle,

$$|g(z)| \leq \frac{1}{r^2}\left[\frac{r}{(1-r)^2} + r\right] \leq \frac{2}{r(1-r)^2} \qquad \text{for} |z| \leq r.$$

When $|z| \leq 1/2$, $r = 1/2$,

$$|g(z)| \leq 16 \leq \frac{16}{(1-|z|)^2}.$$

When $1/2 \leq |z| < 1$, $r = |z|$,

$$|g(z)| \leq \frac{4}{(1-|z|)^2}.$$

So $C = 16$ works. □

**Proposition 5.11.** *Let $\{f_t\}$ be normalized Loewner chain on $I = [a, \infty)$, $f_t(0) = 0$, $f_t'(0) > 0$, $t \in I$. Let $\varphi_{s,t} := f_t^{-1} \circ f_s$ for $a \leq s \leq t$. Then*

$$e^t \varphi_{s,t} \to f_s \qquad \textit{locally uniformly on } \mathbb{D}$$

*as $t \to \infty$ (i.e., along any sequence $t_n \to \infty$).*

*Proof.* Suppose $a \leq s \leq t$, $\varphi_{s,t}(0) = 0$, $\varphi_{s,t}(\mathbb{D}) \subseteq \mathbb{D}$. So

(1) $|\varphi_{s,t}(z)| \leq |z|$ for $z \in \mathbb{D}$ by Schwarz.

Since $\varphi_{s,t}$ is injective on $\mathbb{D}$, $\varphi_{s,t}'(0) = e^{s-t}$, so

(2) $|\varphi_{s,t}(z)| \leq e^{s-t}\dfrac{|z|}{(1-|z|)^2}$ for $z \in \mathbb{D}$ by Koebe.

Since $f_t \circ \varphi_{s,t} = f_s$, $e^{-t} f_t \in \mathcal{S}$, so by Lemma 5.10,

$$|f_t(w) - e^t w| \leq C\frac{e^t|w|^2}{(1-|w|)^2}.$$

Using this for $w = \varphi_{s,t}(z) \in \mathbb{D}$ and $(1) + (2)$, we obtain

$$\begin{aligned} |f_s(z) - e^t\varphi_{s,t}(z)| &= |f_t(\varphi_{s,t}(z)) - e^t\varphi_{s,t}(z)| \\ &\leq C\frac{e^t|\varphi_{s,t}(z)|^2}{(1-|z|)^2} \qquad (|\varphi_{s,t}(z)| \leq |z|) \\ &\leq C\frac{e^t e^{2s-2t}|z|^2}{(1-|z|)^4} = e^{-t}\frac{Ce^{2s}|z|^2}{(1-|z|)^4} \to 0 \end{aligned}$$

locally uniformly on $\mathbb{D}$ as $t \to \infty$. □

$$\{f_t\}\text{: Loewner chain} \qquad \underset{f_s = \lim_{t\to\infty} e^t\varphi_{s,t}}{\overset{\varphi_{s,t} = f_t^{-1}\circ f_s}{\rightleftarrows}} \qquad \varphi_{s,t}\text{: Semi-group}$$

**Theorem 5.12. (Existence and uniqueness for solutions of Loewner-Kufarev equations)** *Let $I = [a,\infty) \subseteq \mathbb{R}$, $V : \mathbb{D} \times I \to \mathbb{C}$ be a.e. defined measurable function such that*

i) *$V(z,\cdot)$ is a.e. defined and measurable for each $z \in \mathbb{D}$,*

ii) *$V(\cdot,t)$ is holomorphic for a.e. $t \in I$,*

iii) *$V(z,t) = -zp(z,t)$ for $z \in \mathbb{D}$, $t \in I$, where $p(\cdot,t) \in \mathcal{P}$.*

*Then there exists a unique normalized Loewner chain $\{f_t\}_{t\in I}$ with $f_t(0) \equiv w_0 \equiv 0$ such that the Loewner-Kufarev equation hold:*

$$\dot{f}_t(z) = -V(z,t)f_t'(z) \qquad \text{for} \quad z \in \mathbb{D}, \ \text{a.e. } t \in I. \tag{18}$$

*Suppose $g : \mathbb{D} \times I \to \mathbb{C}$ is a function such that*

i) *$g(\cdot,t) \in H(\mathbb{D})$, $g(0,t) = 0$, $g'(0,t) = e^t$ for $t \in I$,*

ii) *$g(z,\cdot)$ is uniform Lipschitz on compact subsets of $\mathbb{D} \times I$,*

iii) *$g$ solves (18), i.e.,*

$$\frac{\partial g}{\partial t}(z,t) = -V(z,t)\frac{\partial g}{\partial z}(z,t)$$

*for each $z \in \mathbb{D}$ and a.e. $t \in I$.*

*Then there exists an entire function $h : \mathbb{C} \to \mathbb{C}$ with $h(0) = 0$, $h'(0) = 1$, such that $g_t = h \circ f_t$ for $t \in I$.*

*Suppose $g$ satisfies the following additional assumption:*

iv) *there exist $r_0 \in (0,1)$ and $C \geq 0$ such that $|g_t(z)| \leq Ce^t$ for $t \in I$, $z \in \overline{B}(0,r_0)$. Then $h = \mathrm{id}_{\mathbb{C}}$ and so $g_t = f_t$ for all $t \in I$.*

*Proof.* We know that there exists a normalized Loewner chain $\{f_t\}$ solving (18). (See Corollary 5.8. Find unique $\varphi_{s,t}(z)$ such that $\varphi_{s,s}(z) = z$, $z \in \mathbb{D}$, $\partial\varphi_{s,t}/\partial t = V(\varphi_{s,t}(z),t)$ for a.e. $t \geq s$. Let $f_s := \lim_{t\to\infty} e^t\varphi_{s,t}$. Then $\varphi_{s,t} = f_t^{-1} \circ f_s$. $\{f_t\}_{t\in I}$ is a Loewner chain solving (18).)

Let $g$ be a function as in hypotheses, $g_t := g(\cdot,t)$.

**Claim.** $g_t \circ \varphi_{s,t} = g_s$ for $a \leq s \leq t$.

Fix $s$. Then for $z \in \mathbb{D}$ and a.e. $t \geq s$. By Proposition 4.12 (iii), $g$ is differentiable for a.e. $t \in I$.

$$\begin{aligned}
\frac{d}{dt}g_t \circ \varphi_{s,t}(z) &= \frac{d}{dt}g(\varphi_{s,t}(z),t) \\
&= \frac{\partial g}{\partial z}(\varphi_{s,t}(z),t) \cdot \frac{\partial\varphi_{s,t}(z)}{\partial t} + \frac{\partial g}{\partial t}(\varphi_{s,t}(z),t) \\
&= g_t' \circ \varphi_{s,t}(z) \cdot V(\varphi_{s,t}(z),t) + \dot{g}_t \circ \varphi_{s,t}(z) \\
&= g_t'(w) \cdot V(w,t) + \dot{g}_t(w) = 0.
\end{aligned}$$

Since $t \mapsto g(\varphi_{s,t}(z),t)$ is local Lipschitz, we have

$$g_t \circ \varphi_{s,t}(z) \equiv \text{const.} \quad \text{in } t \geq s, \ \text{and for fixed } s \in I, z \in \mathbb{D}.$$

For $t = s$,

$$g_s \circ \varphi_{s,s}(z) = g_s(z).$$

The Claim follows. By Claim,

$$g_t \circ \varphi_{s,t} = g_s, \quad \Longleftrightarrow \quad g_t \circ f_t^{-1} = g_s \circ f_s^{-1}$$

on $\Omega_s := f_s(\mathbb{D})$ for $t \geq s$. Note $\bigcup_{t\geq a} \Omega_t = \mathbb{C}$, because $\Omega_t \supseteq B(0, \frac{1}{4}e^t)$ by Koebe. Define

$$h(z) = (g_t \circ f_t^{-1})(z) \qquad \text{if} \quad z \in \Omega_t.$$

Then $h$ is well-defined and holomorphic on $\mathbb{C} = \bigcup_{t\in I} \Omega_t$; hence entire.

By definition, $g_t = h \circ f_t$ for $t \in I$.

$$h(0) = h(f_t(0)) = g_t(0) = 0,$$

and

$$h'(0) \circ f_t'(0) = g_t'(0) \quad \Longrightarrow \quad h'(0)e^t = e^t \quad \Longrightarrow \quad h'(0) = 1.$$

Suppose that $g$ satisfies (iv) in addition, then

$$|g_t(z)| = |h(f_t(z))| \leq Ce^t \qquad \text{for} \quad z \in \overline{B}(0, r_0).$$

By Koebe, $f_t(B(0, r_0)) \supseteq B(0, \frac{1}{4}e^t r_0)$, and so

$$|h(w)| \leq Ce^t, \qquad \text{for} \quad w \in B(0, \frac{1}{4}e^t r_0), \quad t \in I.$$

So there exists $C' \geq 0$ such that

$$|h(w)| \leq C'(1 + |w|), \qquad w \in \mathbb{C}.$$

By Cauchy estimate, $h(w) \equiv aw + b$, $a, b \in \mathbb{C}$. Since $h(0) = 0$, $h'(0) = 1$, we have $b = 0$, $a = 1$, and so $h(w) \equiv w$, i.e., $h = \mathrm{id}_{\mathbb{C}}$.

Suppose $\{\tilde{f}_t\}$ is another normalized Loewner chain with $f_t(0) = 0$, $t \in I$, solving (18). Then

$$|\tilde{f}_t(z)| \leq e^t \frac{|z|}{(1 - |z|)^2}, \qquad z \in \mathbb{D}, t \in I,$$

by Koebe, and so

$$|\tilde{f}_t(z)| \leq 2e^t, \qquad |z| \leq \frac{1}{2}, t \in I,$$

i.e., (iv) is true. Moreover, (i)–(iii) are laos true and so $\tilde{f}_t = f_t$ for all $t \in I$, i.e., there exists a unique normalized Loewner chain solving (18). □

**Remark 5.13.** It is likely that the second part of Theorem 5.12 can be proved under weaker regularity assumptions, e.g., namely that $g(\cdot, t) \in H(\mathbb{D})$ for each $t \in I$, and $g(z, \cdot)$ is absolutely continuous on compact $J \subseteq I$ for each $z \in \mathbb{D}$. It is not clear that under those hypotheses $g$ is differentiable for a.e. $(z, t) \in \mathbb{D} \times I$, not even local boundedness is clear!

Figure 19: The Loewner triangle

Recent papers by Bracci, Contreras, Diaz-Madrigal, et.al.

**Theorem 5.14.** *Let $f \in H(\mathbb{D})$, $f'(z) \neq 0$ for $z \in \mathbb{D}$, and*

$$(1-|z|^2)\left|z\frac{f''(z)}{f'(z)}\right| \le 1 \qquad \text{for} \quad z \in \mathbb{D}. \tag{19}$$

*Then $f$ is univalent on $\mathbb{D}$ (injective and holomorphic).*

*Conversely, if $f$ is univalent on $\mathbb{D}$, then $f'(z) \neq 0$ for $z \in \mathbb{D}$, and*

$$(1-|z|^2)\left|z\frac{f''(z)}{f'(z)}\right| < 6 \qquad \text{for} \quad z \in \mathbb{D}.$$

*Proof.* I. Suppose first that $f$ is univalent on $\mathbb{D}$. Wlog $f(0) = 0$, $f'(0) = 1$, so $f \in \mathcal{S}$. Then $f'(z) \neq 0$ for $z \in \mathbb{D}$, and by Lemma 1.6,

$$\left|(1-|z|^2)\frac{f''(z)}{f'(z)} - 2\overline{z}\right| \le 4 \qquad \text{for} \quad z \in \mathbb{D}.$$

Hence,

$$(1-|z|^2)\left|z\frac{f''(z)}{f'(z)}\right| \le 4|z| + 2|z|^2 < 6 \qquad \text{for} \quad z \in \mathbb{D}.$$

II. Suppose now that $f$ satisfies the hypotheses of the first part. Wlog $f(0) = 0$, $f'(0) = 1$. Define

$$\begin{aligned} &f(z,t) := f(e^{-t}z) + (e^t - e^{-t})zf'(e^{-t}z), \qquad z \in, t \in I := [0,\infty),\\ &f_t(z) := f(z,t). \end{aligned}$$

Then $f(\cdot,t) \in H(\mathbb{D})$, $t \in I$, and $f(z,\cdot) \in C^1[0,\infty)$, $z \in \mathbb{D}$.

$$\begin{aligned} \frac{\partial f}{\partial t}(z,t) &= -e^{-t}zf'(e^{-t}z) + (e^t + e^{-t})zf'(e^{-t}z) - (e^t - e^{-t})z^2e^{-t}f''(e^{-t}z)\\ &= e^tzf'(e^{-t}z) - (e^t - e^{-t})z^2e^{-t}f''(e^{-t}z)\\ &= e^tzf'(e^{-t}z)\left[1 - (1-e^{-2t})\frac{e^{-t}zf''(e^{-t}z)}{f'(e^{-t}z)}\right]. \end{aligned}$$

So

$$\left|\frac{\partial f}{\partial t}(z,t)\right| \le M(r,T) \qquad \text{for} \quad |z| \le r < 1,\ 0 \le t \le T.$$

Hence $f \in HL(\mathbb{D} \times I)$.

$$\begin{aligned} \frac{\partial f}{\partial z}(z,t) &= e^{-t}f'(e^{-t}z) + (e^t - e^{-t})\Big[f'(e^{-t}z) + ze^{-t}f''(e^{-t}z)\Big]\\ &= e^tf'(e^{-t}z) + (e^t - e^{-t})ze^{-t}f''(e^{-t}z)\\ &= e^tf'(e^{-t}z)\left[1 + (1-e^{-2t})\frac{e^{-t}zf''(e^{-t}z)}{f'(e^{-t}z)}\right]. \end{aligned}$$

Denote $w = e^{-t}z$. Then $|w| < e^{-t} \le 1$.

$$\left|(1-e^{-2t})\frac{e^{-t}zf''(e^{-t}z)}{f'(e^{-t}z)}\right| < (1-|w|^2)\left|\frac{wf''(w)}{f'(w)}\right| \le 1$$

So $\partial f(z,t)/\partial z \neq 0$. Define

$$V(z,t) := -\frac{\dot{f}(z,t)}{f'(z,t)} = -zp(z,t),$$

where

$$p(z,t) = \frac{1 - B(z,t)}{1 + B(z,t)}, \qquad B(z,t) = (1 - e^{-2t})\frac{e^{-t}zf''(e^{-t}z)}{f'(e^{-t}z)}.$$

For each $t \in I$, $B(\cdot,t) \in H(\mathbb{D})$, $B \in C(\mathbb{D} \times I)$, $|B(z,t)| < 1$ for $(z,t) \in \mathbb{D} \times I$, and $B(0,t) \equiv 0$ for all $t \in I$. Then $p(0,t) \equiv 1$ and $\operatorname{Re} p(\cdot,t) \geq 0$ for all $t \in I$, i.e., $p(\cdot,t) \in \mathcal{P}$, or $V$ is a "Herglotz vector field".

$$\dot{f}(z,t) = -V(z,t)f'(z,t).$$

So $f_t = f(\cdot,t)$ solved the Loewner-Kufarev equation.

There exist $M \geq 0$ such that $|f(z)| \leq M$, $|f'(z)| \leq M$ for $|z| \leq 1/2$. Then

$$|f_t(z)| \leq |f(e^{-t}z)| + e^t|z||f'(e^{-t}z)| \leq M(1 + e^t) \leq 2Me^t, \quad \text{for } t \geq 0.$$

By Theorem 5.12, $\{f_t\}_{t\in[0,\infty)}$ is a Loewner chain, so $f_t$ is univalent for $t \geq 0$. In particular, $f_0 = f$ is univalent. □

# 6 Variants and special cases of the Loewner-Kufarev equations

**6.1. Slit domains**

Let $\gamma : [a,\infty] \to \hat{\mathbb{C}}$ be simple path ending at $\infty$ such that $0 \notin \gamma[a,\infty]$, $\gamma(\infty) = \infty$. Let $\Omega_t = \mathbb{C} \setminus \gamma([t,\infty))$ be simply connected domains. Then $\{\Omega_t\}$ is a geometric Loewner chain. Let $f_t : \mathbb{D} \to \Omega_t$ be the unique conformal map such that $f_t(0) = 0$, $f_t'(0) > 0$. Then $\{f_t\}$ is a Loewner chain. By a homeomorphic reparametrization of time we may wlog assume that $\{f_t\}$ is a normalized Loewner chain, i.e., $f_t'(0) = e^t$, $t \in I$ (cf. Lemma 4.7).

Figure 20: Slit Loewner chain

For $a \leq s < t < \infty$, $\gamma([s,t)) \subseteq \Omega_t$, $\lim_{s'\to t^-} \gamma(s') = \gamma(t) \in \partial\Omega_t$. Hence, by Corollary 2.20,

$$\lambda(t) := \lim_{s'\to t^-} f_t^{-1}(\gamma(s')) \in \partial\mathbb{D} \quad \text{exists.}$$

Denote

$$J_{s,t} = f_t^{-1}([s,t)) \subseteq \mathbb{D}, \qquad \bar{J}_{s,t} = J_{s,t} \cup \{\lambda(t)\}.$$

Since $\hat{\mathbb{C}} \setminus \Omega_t = \gamma([a,\infty])$ is locally connected (w.r.t. chordal metric), $f_t$ has a continuous extension $f : \to \hat{\mathbb{C}}$ (cf. Theorem 2.1 and Remark 2.6). Then

$$f_t(\lambda(t)) = \lim_{s'\to t^-} f_t(f_t^{-1}(\gamma(s')) = \lim_{s'\to t^-} \gamma(s') = \gamma(t).$$

So $f_t(\lambda(t)) = \gamma(t)$. $\lambda(t)$ is uniquely determined by this equation (cf. Proposition 2.7).

Let $\varphi_{s,t} = f_t^{-1} \circ f_s$. $\varphi_{s,t}$ is a conformal map of $\mathbb{D}$ onto the slit domain $\mathbb{D} \setminus J_{s,t} =: U_{s,t}$. $\partial U_{s,t} = \bar{J}_{s,t} \cup \partial\mathbb{D}$ is locally connected, so by Theorem 2.1, $\varphi_{s,t}$ has a continuous extension $\varphi_{s,t} : \overline{\mathbb{D}} \to \overline{U}_{s,t}$. As in Example 2.15, one shows that there exists an open arc $I_{s,t} \subseteq \partial\mathbb{D}$ such that

$$\varphi_{s,t}^{-1}(J_{s,t}) = I_{s,t}. \qquad \text{(cf. Proposition 2.7)}$$

Then $\lambda(s) \in I_{s,t}$, $\varphi_{s,t}(\lambda(s)) \in J_{s,t}$.

Figure 21:

**Lemma 6.2.** *Fix $T \in [a,\infty)$. Then there exists a distortion function $\omega : (0,\infty) \to (0,\infty)$, $\omega(\delta) \to 0$ as $\delta \to 0^+$ such that*
i) $\operatorname{diam}(J_{s,t}) \le \omega(|s-t|)$,
ii) $\operatorname{diam}(I_{s,t}) \le \omega(|s-t|)$, *for $a \le s \le t \le T$.*

*Proof.* By uniform continuous of $\gamma$ on $[a,T]$ it follows that

$$\operatorname{diam}(\gamma[s,t)) \le \omega_1(|s-t|), \qquad a \le s < t \le T,$$

for some distortion function $\omega_1$ (here and in what follows, we assume the distortion function $\omega(\delta)$ is monotonically increasing as $\delta$ increasing).

Set $g_t = f_t^{-1}$. By Theorem 2.17,

$$\begin{aligned}\operatorname{diam}(J_{s,t}) &= \operatorname{diam}(g_t(\gamma[s,t))) \\ &\le \omega_2\Big(\frac{\operatorname{diam}(\gamma[s,t))}{f_t'(0)}\Big) \le \omega_2(e^{-a}\operatorname{diam}(\gamma[s,t))) \le \omega_3(|s-t|).\end{aligned}$$

So $\operatorname{diam}(J_{s,t})$ is uniformly small if $s<t$ are close in $[a,T]$. Wlog, assume $s<t$ are so close that $\operatorname{diam}(J_{s,t}) < 1/2$.

Let $z_0 := \lambda(t)$, $r = 2\operatorname{diam}(J_{s,t})$. Then $J_{s,t} \subseteq B := B(z_0,r)$ but $0 \notin B(z_0,r)$. So the arc $C \subseteq \mathbb{D} \cap \partial B$ separates 0 and $J_{s,t}$ in $\mathbb{D}$. Then $\tilde{C} = \varphi_{s,t}^{-1}(C)$ separates 0 and $I_{s,t}$ in $\mathbb{D}$. Hence, by Theorem 2.17,

$$\operatorname{diam}(I_{s,t}) \le \omega_4(\operatorname{diam}(\tilde{C})) \le \omega_5\Big(\frac{\operatorname{diam}(C)}{\varphi_{s,t}'(0)}\Big) \le \omega_5(e^{t-s}\operatorname{diam}(C)) \le \omega_6(J_{s,t}) \le \omega_7(|s-t|). \quad \square$$

Let $\Omega \subseteq \hat{\mathbb{C}}$ be open, $f : \Omega \to \mathbb{C}$ be holomorphic ($f$ holomorphic at $\infty$ if $z \mapsto f(1/z)$ holomorphic at 0). Define

$$\begin{aligned}\operatorname{Cl}(f,\Omega) = \{w \in \hat{\mathbb{C}} : &\text{there exists sequence } \{z_n\} \text{ in } \Omega \\ &\text{such that } z_n \to z_0 \in \partial\Omega \text{ and } f(z_n) \to w\},\end{aligned}$$

the set of cluster values of $f$ on $\Omega$.

**Proposition 6.3.** *Let $\Omega \subsetneq \hat{\mathbb{C}}$, $f : \hat{\mathbb{C}} \to \mathbb{C}$ holomorphic. Then*
i) $\sup_{z\in\Omega} |f(z)| = \sup\{|w| : w \in \operatorname{Cl}(f,\Omega)\} \in [0,\infty]$ *(a version of maximum principle),*
ii) *if* $\operatorname{Cl}(f,\Omega) \subseteq \mathbb{C}$, *then* $\operatorname{osc}(f,\Omega) := \sup\{|f(z_1) - f(z_2)| : z_1, z_2 \in \Omega\} = \sup\{|w_1 - w_2| : w_1, w_2 \in \operatorname{Cl}(f,\Omega)\} = \operatorname{diam}(\operatorname{Cl}(f,\Omega))$.

*Proof.* i) The proof is standard. "$\ge$" is clear. For "$\le$": there exists a sequence $\{z_n\}$ in $\Omega$ such that

$$|f(z_n)| \to M := \sup_{z\in\Omega} |f(z)|, \qquad \text{as} \quad n \to \infty.$$

Wlog, assume $z_n \to z_0 \in \overline{\Omega}$, $f(z_n) \to w \in \hat{\mathbb{C}}$ with $M = |w|$.

*Case* 1: $z_0 \in \partial\Omega$. Then $w \in \operatorname{Cl}(f,\Omega)$, and $M = |w|$. We have done!

*Case* 2: $z_0 \in \Omega$. Then $|f|$ attains a maximum at $z_0$. By the maximum principle, $f \equiv w$ on the component $U$ of $\Omega$ with $z_0 \in U$. Then we also have $w \in \mathrm{Cl}(f, \Omega)$ and $M = |w|$.

ii) "$\geq$" is clear. For "$\leq$": Let $z_1, z_2 \in \Omega$ be arbitrary. Consider the map $z \mapsto f(z) - f(z_2)$. It is holomorphic on $\Omega$, so by i) there exists $w_1 \in \mathrm{Cl}(f, \Omega) \subseteq \mathbb{C}$ such that

$$|f(z_1) - f(z_2)| \leq |w_1 - f(z_2)|.$$

Applying the same argument to $z \mapsto w_1 - f(z)$, we find $w_2 \in \mathrm{Cl}(f, \Omega) \subseteq \mathbb{C}$ such that

$$|f(z_1) - f(z_2)| \leq |w_1 - f(z_2)| \leq |w_1 - w_2|.$$

The result follows. $\square$

**Lemma 6.4.** *Setup as in 6.1, $T \in [a, \infty)$. Then there exists a distortion function $\omega$ such that*

$$|\varphi_{s,t}(z) - e^{t-s}z| \leq \omega(|s-t|), \qquad \textit{for} \quad z \in \overline{\mathbb{D}},\ 0 \leq s \leq t \leq T,\ |s-t| \textit{ small.}$$

*Proof.* Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, $R(z) = 1/\overline{z}$, be the reflection w.r.t. $\partial\mathbb{D}$. Let $J^*_{s,t} = R(J_{s,t})$. By the Schwarz reflection principle, $\varphi_{s,t}$ has an extension to a conformal map

$$\varphi_{s,t} : \Omega := \hat{\mathbb{C}} \setminus \{\bar{I}_{s,t}\} \to \Omega' := \hat{\mathbb{C}} \setminus \{\bar{J}_{s,t} \cup J^*_{s,t}\}.$$

by

$$\varphi_{s,t}(z) = R(\varphi_{s,t}(R(z))) \qquad \text{for} \quad |z| > 1.$$

Near 0, $\varphi_{s,t}$ has the expansion

$$\varphi_{s,t}(z) = e^{s-t}z + a_2 z^2 + \cdots .$$

So near $\infty$,

$$\varphi_{s,t}(z) = e^{t-s}z + c_0 + \frac{c_1}{z} + \cdots ,$$

which implies that $\varphi_{s,t}$ has a $1^{st}$ order pole at $\infty$. Let

$$f(z) = \varphi_{s,t}(z) - e^{t-s}z, \qquad \text{for} \quad z \in \Omega.$$

Then $f : \Omega \to \mathbb{C}$ is holomorphic on $\Omega$ with removable singularity at $\infty$.

$$\begin{aligned}\mathrm{Cl}(f, \Omega) &= \{w \in \mathbb{C} : \text{there exists } \{z_n\} \text{ in } \Omega, z_n \to z_0 \in \partial\Omega = \bar{I}_{s,t}, f(z_n) \to w\} \\ &\subseteq A + B := \{a + b : a \in A, b \in B\},\end{aligned}$$

where $A = \bar{J}_{s,t} \cup J^*_{s,t}$, $B = \{-e^{t-s}z_0 : z_0 \in \bar{I}_{s,t}\}$. Note that $f(0) = 0$. By Proposition 6.3,

$$\begin{aligned}\sup_{z \in \overline{\mathbb{D}}} |\varphi_{s,t}(z) - e^{t-s}z| &= \sup_{z \in \mathbb{D}} |f(z)| = \sup_{z \in \mathbb{D}} |f(z) - f(0)| \\ &\leq \mathrm{osc}(f, \Omega) \leq \mathrm{diam}(\mathrm{Cl}(f, \Omega)) \leq \mathrm{diam}(A) + \mathrm{diam}(B).\end{aligned}$$

If $|s-t|$ small, $\mathrm{diam}(J_{s,t})$ is small,

$$\mathrm{diam}(J^*_{s,t}) \lesssim \mathrm{diam}(J_{s,t}) \leq \omega_1(|s-t|).$$

So $\operatorname{diam}(A) \le \omega_2(|s-t|)$.

If $|s-t|$ small, $e^{t-s} \lesssim 1$, and

$$\operatorname{diam}(B) \lesssim \operatorname{diam}(I_{s,t}) \le \omega_3(|s-t|).$$

Hence,

$$\sup_{z\in\mathbb{D}} |\varphi_{s,t}(z) - e^{t-s}z| \le \operatorname{diam}(A) + \operatorname{diam}(B) \le \omega(|s-t|). \qquad \square$$

**Corollary 6.5.** *$\lambda$ (as in 6.1) is a continuous function on $[a,\infty)$.*

*Proof.* Let $a \le s < t \le T$ for any given $T$. Then $\lambda(t), \varphi_{s,t}(\lambda(s)) \in \bar{J}_{s,t}$. We have

(1) $|\lambda(t) - \varphi_{s,t}(\lambda(s))| \le \operatorname{diam}(J_{s,t}) \le \omega_1(|s-t|)$,

(2) $|\varphi_{s,t}(\lambda(s)) - e^{t-s}\lambda(s)| \le \omega_2(|s-t|)$, (Lemma 6.4)

(3) $|e^{t-s}\lambda(s) - \lambda(s)| \le |e^{t-s} - 1| \le \omega_3(|s-t|)$.

By (1) – (3), $|\lambda(t) - \lambda(s)| \le \omega(|s-t|)$. So $\lambda$ is continuous on $[0,T]$. Since $T$ is arbitrary, $\lambda$ is continuous on $[0,\infty)$. $\square$

**Theorem 6.6. (Loewner equation for slit mappings)** *Let $\{f_t\}$ be a Loewner chain generated by a slit (as in 6.1). Then*

$$\dot{f}_t(z) = -V(z,t)f_t'(z) \qquad \text{for a.e. } t \in [a,\infty),\ z \in \mathbb{D},$$

*where*

$$V(z,t) = -z\frac{\lambda(t)+z}{\lambda(t)-z}, \qquad (z,t) \in \mathbb{D} \times I.$$

*Here, $\lambda : I = [a,\infty) \to \partial\mathbb{D}$ is continuous.*

*Proof.* Let $\varphi_{s,t} = f_t^{-1} \circ f_s$. We know from Theorem 4.13 that $\{f_t\}$ satisfies the Loewner-Kufarev equation with

$$V(z,t) = \lim_{\varepsilon\to 0} \frac{\varphi_{t,t+\varepsilon}(z) - z}{\varepsilon}, \qquad z \in \mathbb{D},\ \text{a.e. } t \in I.$$

For $a \le s < t < \infty$, define

$$\Phi_{s,t}(z) := \log\Big(\frac{z}{\varphi_{s,t}(z)}\Big) = (t-s) + \cdots$$

which is holomorphic in $\mathbb{D}$ (cf. (10) in the Proof of Lemma 4.10). Actually, $z \mapsto z/\varphi_{s,t}(z)$ has a zero-free continuous extension to $\overline{\mathbb{D}}$; hence this function has a continuous logarithm on $\overline{\mathbb{D}}$ (uniquely determined by a point normalization). Hence, $\Phi_{s,t}$ has a continuous extension to $\overline{\mathbb{D}}$. By the Schwarz formula

$$\Phi_{s,t}(z) = i\operatorname{Im}\Phi_{s,t}(0) + \frac{1}{2\pi}\int_0^{2\pi} \frac{\zeta+z}{\zeta-z}\operatorname{Re}\Phi_{s,t}(\zeta)|d\zeta|,$$

where $\zeta = e^{it}$, $|d\zeta| = dt$. Note $\operatorname{Im}\Phi_{s,t}(0) = 0$,

$$\operatorname{Re}\Phi_{s,t}(\zeta) = \log\left|\frac{\zeta}{\varphi_{s,t}(\zeta)}\right| = \log\left|\frac{1}{\varphi_{s,t}(\zeta)}\right| \ge 0, \qquad \text{for} \quad \zeta \in \partial\mathbb{D},$$

and

$$|\varphi_{s,t}(\zeta)| = 1 \qquad \text{for} \quad \zeta \in \partial\mathbb{D} \setminus I_{s,t}.$$

So $\operatorname{Re}\Phi_{s,t}(\zeta)$ is supported on $\bar{I}_{s,t} \ni \lambda(s)$.

Since

$$t - s = \Phi_{s,t}(0) = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\Phi_{s,t}(\zeta)|d\zeta|,$$

We can define a probability measure $\mu_{s,t}$ on $\partial\mathbb{D}$ by

$$d\mu_{s,t}(\zeta) = \frac{1}{2\pi(t-s)} \operatorname{Re}\Phi_{s,t}(\zeta)|d\zeta|.$$

Then $\operatorname{supp}(\mu_{s,t}) \subseteq \bar{I}_{s,t} \ni \lambda(s)$. Fix $s$, and let $t = s+\varepsilon$, $\varepsilon \to 0^+$. Then $\operatorname{diam}(I_{s,s+\varepsilon}) \to 0$ (Lemma 6.2). Hence,

$$\mu_{s,s+\varepsilon} \xrightarrow{w^*} \delta_{\lambda(s)} \quad \text{(Dirac mass at } \lambda(s)) \quad \text{as} \quad \varepsilon \to 0^+.$$

i.e.,

$$\int_{\partial\mathbb{D}} h(\zeta)d\mu_{s,s+\varepsilon}(\zeta) \to \int_{\partial\mathbb{D}} h(\zeta)d\delta_{\lambda(s)} = h(\lambda(s)), \quad \text{for} \quad h \in C(\partial\mathbb{D}).$$

So

$$\begin{aligned}
\lim_{\varepsilon\to 0^+} \frac{\Phi_{s,s+\varepsilon}(z)}{\varepsilon} &= \lim_{\varepsilon\to 0^+} \int_{\partial\mathbb{D}} \frac{\zeta+z}{\zeta-z} d\mu_{s,s+\varepsilon}(\zeta) \\
&= \int_{\partial\mathbb{D}} \frac{\zeta+z}{\zeta-z} d\delta_{\lambda(s)} = \frac{\lambda(s)+z}{\lambda(s)-z}, \qquad \text{for all } s \in I, z \in \mathbb{D}.
\end{aligned}$$

On the other hand, $\varphi_{s,t}(z) = -z\exp(-\Phi_{s,t}(z))$. So

$$\begin{aligned}
V(z,t) &= \lim_{\varepsilon\to 0^+} \frac{\varphi_{s,s+\varepsilon}(z) - z}{\varepsilon} = \lim_{\varepsilon\to 0^+} z\frac{\exp(-\Phi_{s,s+\varepsilon}(z)) - 1}{\varepsilon} \\
&= z\frac{\partial}{\partial\varepsilon}\exp(-\Phi_{s,s+\varepsilon}(z))\Big|_{\varepsilon=0} = -z\exp(0)\frac{\partial\Phi_{s,s+\varepsilon}(z)}{\partial\varepsilon}\Big|_{\varepsilon=0} \\
&= -z\frac{\lambda(s)+z}{\lambda(s)-z}.
\end{aligned}$$

Here, we have used the fact

$$\lim_{\varepsilon\to 0^+} \Phi_{s,s+\varepsilon}(z) = \lim_{\varepsilon\to 0^+} \varepsilon \cdot \frac{\Phi_{s,s+\varepsilon}(z)}{\varepsilon} = 0. \qquad \square$$

**Example 6.7.** If $\lambda(t) \equiv 1$, then

$$f_t(z) = \frac{e^t z}{(1+z)^2}.$$

In fact,

$$\dot{f}_t(z) = \frac{e^t z}{(1+z)^2}, \qquad f_t'(z) = e^t\frac{1-z}{(1+z)^3}.$$

So

$$\dot{f}_t(z) = z\frac{1+z}{1-z}f_t'(z).$$

**Example 6.8.** *Stationary solutions of the Loewner-Kufarev equation.*

Let $f \in H(\mathbb{D})$ with $f(0) = 0$, $f'(0) = 1$. Suppose that $f_t(z) = a(t)f(z)$ is a normalized Loewner chain. Then $f_t'(0) = a(t)f'(0) = a(t) = e^t$. So

$$f_t(z) = e^t f(z).$$

Note

$$\dot{f}_t(z) = e^t f(z), \qquad f_t'(z) = e^t f'(z).$$

The Loewner-Kufarev equation implies

$$\dot{f}_t(z) = e^t f(z) = -V(z,t) f_t'(z) = zp(z,t)e^t f'(z),$$

where

$$p(z,t) = \frac{f(z)}{zf'(z)} \in \mathcal{P}. \qquad \text{(0 is a removable singularity)}$$

So

$$\operatorname{Re} p(z,t) > 0 \Longleftrightarrow \operatorname{Re}\Big(\frac{f(z)}{zf'(z)}\Big) > 0 \Longleftrightarrow \operatorname{Re}\Big(\frac{zf'(z)}{f(z)}\Big) > 0.$$

**Theorem 6.9.** *Let $f \in H(\mathbb{D})$, $f(0) = 0$, $f'(0) = 1$. TFAE.*

i) $\operatorname{Re}\Big(z\frac{f'(z)}{f(z)}\Big) > 0$ *(has removable singularities by assumption),*

ii) *$f \in \mathcal{S}$ and $\Omega = f(\mathbb{D})$ is starlike with respect to 0, i.e., $[0, w] \subseteq \Omega$ for all $w \in \Omega$.*

*Proof.* i) $\Rightarrow$ ii): By Example 6.8, $F(z,t) = f_t(z) - e^t f(z)$ solves the Loewner-Kufarev equation. $F$ is $C^\infty$-smooth on $\mathbb{R} \times \mathbb{D}$ and $|f_t(z)| \leq Ce^t$ for $t \in \mathbb{R}$, $z \in \overline{B}(0, 1/2)$; $f_t'(0) = e^t$, $t \in \mathbb{R}$. Hence, $\{f_t\}$ is a normalized Loewner chain; so $f = f_0$ is a conformal map and

$$\Omega_t := f_t(\mathbb{D}) = e^t\Omega \subseteq \Omega_0 = \Omega$$

for all $t < 0$. So $f \in \mathcal{S}$ and $\Omega$ is starlike w.r.t. 0.

ii) $\Rightarrow$ i): If $f \in \mathcal{S}$ and $\Omega$ is starlike w.r.t. 0, then $\{\Omega_t\}_{t \in \mathbb{R}}$ with $\Omega_t = e^t\Omega$ forms a geometric Loewner chain, corresponding to the analytic Loewner chain $\{f_t\}_{t\in\mathbb{R}}$ with $f_t(z) = e^t f(z)$. Hence, $\operatorname{Re}(zf'(z)/f(z)) > 0$ by Example 6.8. □

# 7 The radial and chordal versions of the Loewner-Kufarev equation

**7.1. Radial Loewner chains (disk version of Loewner chain).**

Let $I = [0, b]$ with $b \in (0, \infty]$. The sequence of regions $\{\Omega_t\}_{t\in I}$ is called a *(geometric) radial Loewner chain* if

i) $\Omega_t \subseteq \mathbb{D}$ is a simply connected region with $0 \in \Omega_t$ for $t \in I$,

ii) $\Omega_0 = \mathbb{D}$,

iii) $\Omega_s \supsetneq \Omega_t$ for $s < t$, $s, t \in I$,

iv) $\{\Omega_t\}$ is continuous in $t$ in sense of kernel convergence with respect to $w_0 = 0$.

If $f_t : \mathbb{D} \longleftrightarrow \Omega_t$ be the unique conformal map with $f_t(0) = 0$, $f_t'(0) > 0$, then $\{f_t\}_{t\in I}$ is the corresponding *(analytic) radial Loewner chain.* It is normalized if $f_t'(0) = e^{-t}$ for $t \in I$.

Simplest situation: $\Omega_t = \mathbb{D} \setminus [1/t, 1)$, "a radius grows out of $\partial\mathbb{D}$ towards 0".

Study of radial Loewner chain can be reduced to whole plane version. If $\{\Omega_t\}_{t\in[0,b]}$ is a radial Loewner chian, define

$$\tilde{\Omega}_t = \begin{cases} \Omega_{-t} & \text{for} \quad t \in [-b, 0], \\ e^t\mathbb{D} & \phantom{for} \quad t \geq 0. \end{cases}$$

(continuity clear, also at $t = 0$.) Then $\{\Omega_t\}_{t\in[-b,\infty)}$ is a (whole plane) Loewner chain. If the $\{\Omega_t\}$ is normalized (i.e., the corresponding analytic Loewner chain is), then $\{\tilde{\Omega}_t\}$ is normalized. $\{\Omega_t\}$ can be obtained from $\{\tilde{\Omega}_t\}$ by "time reversed and restriction of time interval. So the regularity theory for whole plane Loewner chains remains valid in radial case, in particular, if $\{f_t\}_{t\in[0,b]}$ is a normalized radial Loewner chain, then

$$\dot{f}_t(z) = V(z,t) f_t'(z) \qquad \text{for} \quad \text{a.e. } t \in I, \text{ all } z \in \mathbb{D},$$

where $V$ is a Herglotz vector field (radial Loewner-Kufarev equation). Note the sign change in comparison to Loewner-Kufarev equation due to time reversal!.

**7.2. Radial Loewner chains generated by slits.**

Let $\gamma : [0,b] \to \mathbb{C}$ be a simple path, $\gamma(0) = 1$, $\gamma(t) \in \mathbb{D}$, $t \in (0,b]$, $0 \notin \gamma[0,b]$. Let $\Omega_t = \mathbb{D} \setminus \gamma([0,t]) \subseteq \mathbb{D}$ be a simply connected region with $0 \in \Omega_t$, $\Omega_0 = \mathbb{D}$, $\Omega_t \subseteq \Omega_s$, $t < s$. Then $\{\Omega_t\}_{t\in[0,b]}$ is a geometric radial Loewner chain. We can assume that the corresponding analytic radial Loewner chain $\{f_t\}$ is normalized: $f_t(0) = 0$, $f_t'(0) = e^{-t}$.

Figure 22: Radial Loewner chain and corresponding maps

$$f_t(\lambda(t)) = \gamma(t), \qquad \varphi_{s,t}(\lambda(s)) \in J_{s,t}.$$

**Lemma 6.2.** *$J_{s,t}$, $I_{s,t}$ are uniformly small if $|s - t|$ is small.*

**Lemma 6.4.** *$\varphi_{s,t}$ is uniformly close to* $\mathrm{id}_\mathbb{C}$ *if $|s - t|$ is small.*

**Corollary 6.5.** *$|\lambda(s) - \lambda(t)|$ is uniformly small if $|s - t|$ is small. $\lambda$ is continuous.*

Proof of Theorem 6.6 shows

$$\dot{f}_t(z) = -z\frac{\lambda(t) + z}{\lambda(t) - z} f_t'(z), \qquad (z,t) \in \mathbb{D} \times [0,b].$$

**7.3. Idea of chordal Loewner chains.**

Figure 23: Conformal maps

Let $f_t : \mathbb{D} \to \Omega_t$ be conformal maps. We want to normalize conformal maps at boundary point, say $1 \in \partial\mathbb{D}$. Meaningless, unless we have additional assumptions:

$\Omega_t \subseteq \mathbb{D}$ such that $B(1, r(t)) \cap \mathbb{D} \subseteq \Omega_t$, $\Omega_t \supseteq \Omega_s$ as $t < s$.

Figure 24: Additional assumptions for $\Omega_t$

Simplest situation: $\Omega_t = \mathbb{D} \setminus (-1, 1 - t]$, $t \in [0, 2]$. (figure)

Mostly, one switches to upper-half plane $\mathbb{H} = \{w \in \mathbb{C} : \operatorname{Im} w > 0\}$, $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$, and $\mathbb{D} \cup \{1\} \longleftrightarrow \mathbb{H} \cup \{\infty\}$.

**Lemma 7.4.** *Let $\Omega \subseteq \mathbb{D}$ be simply connected region, $g : \Omega \leftrightarrow \mathbb{D}$ be conformal map. Suppose $\zeta \in \partial\mathbb{D} \cap \partial\Omega$ and there exists $r > 0$ such that $\mathbb{D} \cap B(\zeta, r) \subseteq \Omega$. Then $g$ has a holomorphic extension to a neighborhood of $\zeta$ and $g'(\zeta) \neq 0$.*

*Proof.* Wlog, we assume $\zeta = 1$ and there exists an open arc $\alpha \subseteq \partial\mathbb{D}\cap\partial\Omega$ with $1 \in \alpha$. By Wolff's lemma, $g$ has a continuous extension to $\Omega\cup\alpha$. Then $g(\alpha) \subseteq \partial\mathbb{D}$, and $g$ extends to a holomorphic function near $\zeta$. Points in $\mathbb{D}$ near $g(\zeta) \in \partial\mathbb{D}$ have precisely one preimage near $\zeta$, so $g$ is locally injective near $\zeta$ and $g'(\zeta) \neq 0$. □

Note that $f = g^{-1}$ has a locally injective extension to $\eta = g(\zeta) \in \partial\mathbb{D}$.

**Corollary 7.5.** *Let $\Omega \subseteq \mathbb{H}$ be a simply connected region such that $\mathbb{H} \setminus B(0, R) \subseteq \Omega$ for some $R > 0$. Then there exists a unique conformal map $f : \mathbb{H} \leftrightarrow \Omega$ such that $f$ has a holomorphic extension near $\infty$ and*

$$f(z) = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots, \qquad \text{for } z \text{ near } \infty.$$

*Proof.* Existence: By Lemma 7.4, there exists a conformal map $g : \Omega \leftrightarrow \mathbb{H}$ such that $g$ has a holomorphic and locally injective extension to $\infty$ with $g(\infty) \in \hat{\mathbb{R}}$. Post-composition by a Möbius transformation, we may assume $g(\infty) = \infty$. Since $g$ is locally injective, $g$ has the first order pole near $\infty$, and so

$$g(z) = b_1 z + b_0 + \frac{b_{-1}}{z} + \frac{b_{-2}}{z^2} + \cdots,$$

$g(x) \in \hat{\mathbb{R}}$ for $x \in \mathbb{R}$ near $\infty$; so

$$b_1 = \lim_{x\in\mathbb{R}\to\infty} \frac{g(x)}{x} \in \mathbb{R};$$
$$b_0 = \lim_{x\in\mathbb{R}\to\infty} g(x) - b_1 x \in \mathbb{R}.$$

Since $\operatorname{Im} g(ix) > 0$ for $x \in \mathbb{R}$, so

$$b_1 = \operatorname{Re} b_1 = \lim_{x\to+\infty} \operatorname{Re}\Big(\frac{g(ix)}{ix}\Big) \geq 0,$$

so $b_1 > 0$. Then $\varphi(w) = (w - b_0)/b_1$ preserves $\mathbb{H}$, and $\tilde{g} := \varphi \circ g$ is a conformal map of $\Omega$ onto $\mathbb{H}$ with

$$\tilde{g}(z) = z + \frac{\tilde{b_{-1}}}{z} + \cdots, \qquad \text{near} \quad \infty.$$

Let $f := \tilde{g}^{-1}$. Then $f : \mathbb{H} \leftrightarrow \Omega$ is a conformal map, holomorphic near $\infty$, and

$$f(z) = z + \frac{a_1}{z} + \frac{z_2}{z^2} + \cdots \qquad \text{for } z \text{ near } \infty.$$

Uniqueness: Suppose $f_1, f_2 : \mathbb{H} \leftrightarrow \Omega$ are two conformal maps, holomorphic near $\infty$, and

$$f_1(z) = z + o(1), \qquad f_2(z) = z + o(1).$$

Then $\varphi := f_2 \circ f_1^{-1} : \mathbb{H} \leftrightarrow \mathbb{H}$ is a conformal map, hence a Höbius transformation with $\varphi(\mathbb{H}) = \mathbb{H}$.

$$\varphi(z) = \frac{az + b}{cz + d}, \qquad a, b, c, d \in \mathbb{R},\ ad - bc > 0.$$

Moreover, $\varphi(\infty) = \infty$, so $\varphi(z) = az + b$, $a > 0$, $b \in \mathbb{R}$. $\varphi(z) = z + o(1)$, so $a = 1$, $b = 0$, and $\varphi = \mathrm{id}_{\hat{\mathbb{C}}}$. Hence, $f_1 = f_2$. □

**Theorem 7.6. (Herglotz representation for positive harmonic functions)** a) *(disk version) Let $h : \mathbb{D} \to (0,\infty)$ be a positive harmonic function. Then there exists a unique positive measure $\mu$ on $\partial\mathbb{D}$ with $0 < \mu(\partial\mathbb{D}) < \infty$ such that*

$$h(z) = \int_{\partial\mathbb{D}} \operatorname{Re}\Big(\frac{\zeta + z}{\zeta - z}\Big) d\mu(\zeta), \qquad z \in \mathbb{D}.$$

b) *(half-plane version) Let $h : \mathbb{H} \to (0,\infty)$ be a positive harmonic function. Then there exist a unique constant $a \geq 0$ and a unique positive measure $\nu$ on $\mathbb{R}$ such that*

$$0 < a + \int_{\mathbb{R}} \frac{1}{1+t^2} d\nu(t) < \infty,$$

*and*

$$h(z) = a \cdot \operatorname{Im} z + \int_{\mathbb{R}} \operatorname{Im}\Big(\frac{1}{t-z}\Big) d\nu(t), \qquad z \in \mathbb{H}.$$

*Proof.* a) $h$ is a positive harmonic function on $\mathbb{D}$ if and only if there exists a unique $f \in H(\mathbb{D})$ such that $\operatorname{Re} f = h > 0$, $f(0) = h(0) > 0$, if and only if there exists a unique measure $\mu \geq 0$ on $\partial\mathbb{D}$ such that

$$f(z) = \int_{\partial\mathbb{D}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta),$$

with $0 < \mu(\partial D) = f(0) < \infty$. The existence and uniqueness follow.

b) Let $\varphi : \mathbb{D} \cup \{1\} \leftrightarrow \mathbb{H} \cup \{\infty\}$ be conformal map with $\varphi(1) = \infty$, say

$$z = \varphi(w) = i\frac{1+w}{1-w}, \qquad w = \psi(z) = \varphi^{-1}(z) = \frac{z-i}{z+i}.$$

Suppose that $h : \mathbb{H} \to (0,\infty)$ is harmonic, $\Delta h = 0$. Then $g = h \circ \varphi : \mathbb{D} \to (0,\infty)$ is harmonic on $\mathbb{D}$, $\Delta g = 0$. There exists a unique holomorphic function on $\mathbb{D}$ such that $\operatorname{Re} f = g$, $f(0) = g(0) > 0$. By (a),

$$f(w) = a \cdot \frac{1+w}{1-w} + \int_{\partial\mathbb{D}\setminus\{1\}} \frac{\zeta + w}{\zeta - w} d\mu(\zeta), \qquad \text{where } a = \mu(\{1\}) \geq 0.$$

Let $\tau := \varphi_* \mu|_{\partial\mathbb{D}\setminus\{1\}}$ be the measure on $\mathbb{R}$, $\tau(A) = \mu(\varphi^{-1}(A))$ for $A \subseteq \mathbb{R}$.

$$\int_{\mathbb{R}} \rho d\tau = \int_{\partial\mathbb{D}\setminus\{1\}} (\rho \circ \varphi) d\mu, \qquad \rho \in L^1(\tau),$$

$0 < \mu(\partial\mathbb{D}) = a + \tau(\mathbb{R}) < \infty$. $(a, \tau)$ are unique.

Let $\tilde{f}(z) = f(\psi(z))$, $z \in \mathbb{H}$. Since $(1+w)/(1-w) = -iz$,

$$\operatorname{Re}\Big(\frac{1+w}{1-w}\Big) = \operatorname{Re}(-iz) = \operatorname{Im} z.$$

Set $\zeta = (t-i)/(t+i)$, $t \in \mathbb{R} \longleftrightarrow \zeta \in \partial\mathbb{D} \setminus \{1\}$. Then

$$\frac{\zeta + w}{\zeta - w} = -i\Big(\frac{1+tz}{t-z}\Big) = -i\Big(\frac{1+t^2}{t-z} - t\Big),$$

and

$$\operatorname{Re}\Big(\frac{\zeta + w}{\zeta - w}\Big) = \operatorname{Re}\Big(-i\Big[\frac{1+t^2}{z-t}\Big]\Big) = (1+t^2)\operatorname{Im}\Big(\frac{1}{z-t}\Big).$$

Define measure $\nu$ on $\mathbb{R}$ by

$$d\nu(t) = (1+t^2)d\tau(t).$$

Then

$$\int_{\mathbb{R}} \frac{1}{1+t^2} d\nu(t) = \int_{\mathbb{R}} d\tau(t) < \infty,$$

and

$$0 < a + \int_{\mathbb{R}} \frac{1}{1+t^2} d\nu(t) = a + \tau(\mathbb{R}) = \mu(\partial\mathbb{D}) < \infty.$$

Then

$$\tilde{f}(z) = a(-iz) + \int_{\mathbb{R}} (-i)\Big(\frac{1+t^2}{t-z} - t\Big) d\tau(t), \qquad z \in \mathbb{D}.$$

Hence

$$h(z) = \operatorname{Re} \tilde{f}(z) = a \operatorname{Im} z + \int_{\mathbb{R}} \operatorname{Im}\Big(\frac{1}{t-z}\Big) d\nu(t), \qquad z \in \mathbb{D}.$$

Setting $z = x + iy$, $y > 0$, the integral converges since

$$\operatorname{Im}\Big(\frac{1}{t-z}\Big) = \operatorname{Im}\Big(\frac{1}{(t-x)-iy}\Big) = \frac{y}{(x-t)^2+y^2} \lesssim \frac{1}{1+t^2}$$

for $x, y$ fixed, $|t|$ large.

The uniqueness of $(a, \nu)$ is clear. □

**Remark 7.7.** If $g \in H(\mathbb{H})$, $\operatorname{Im} g > 0$. Let $f = -ig$, $g = if$. Then $\operatorname{Re} f > 0$. The proof shows that there exist unique constants $a, b \in \mathbb{R}$, $a \geq 0$, and a Lebesgue finite measure $\tau \geq 0$ on $\mathbb{R}$, such that

$$g(z) = az + b + \int_{\mathbb{R}} \Big(\frac{1+t^2}{t-z} - t\Big) d\tau(t), \qquad z \in \mathbb{H}.$$

**Theorem 7.8. (Julia's Lemma)** *Let $f : \mathbb{H} \to \mathbb{H}$ be holomorphic, and*

$$c := \inf_{z\in\mathbb{H}} \frac{\operatorname{Im} f(z)}{\operatorname{Im} z} \geq 0.$$

*Then*

$$c = \lim_{y\to+\infty} \frac{\operatorname{Im} f(iy)}{y}. \tag{20}$$

*Suppose in addition that $f$ is holomorphic near $\infty$, and has a Laurent expansion of the form*

$$f(z) = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots \tag{21}$$

*near $\infty$. Then $c = 1$ and $a_1 \leq 0$ (so $a_1 \in \mathbb{R}$) with equality iff $f(z) = z$ for $z \in \mathbb{H}$.*

Note: $\operatorname{Im} f(z) \geq \operatorname{Im} z$ for $z \in \mathbb{H}$, and so $f(H_t) \subseteq H_t$ $(t \geq 0)$, where $H_t = \{z \in \mathbb{C} : \operatorname{Im} z > t\}$.

*Proof.* Let $h := \operatorname{Im} f$. $h \geq 0$, $\Delta h = 0$. Wlog, $h > 0$ (otherwise, $f \equiv a \in \mathbb{R}$, claim true). By Theorem 7.6

$$h(z) = a \cdot \operatorname{Im} z + \int_{\mathbb{R}} \operatorname{Im}\Big(\frac{1}{t-z}\Big) d\nu(t),$$

where $a \geq 0$, $\nu \geq 0$ and

$$\tilde{h}(z) = \int_{\mathbb{R}} \operatorname{Im}\Big(\frac{1}{t-z}\Big) d\nu(t) \geq 0, \qquad \text{for } z \in \mathbb{H}.$$

So

$$\frac{h(z)}{\operatorname{Im} z} = a + \frac{\tilde{h}(z)}{\operatorname{Im} z},$$

which implies $c \geq a$. For claim, it suffices to show that

$$\lim_{y \to +\infty} \frac{\tilde{h}(iy)}{y} = 0, \qquad \text{(then } c = a \text{ and (1) true.)}$$

However,

$$\operatorname{Im}\Big(\frac{1}{t-iy}\Big) = \frac{y}{t^2+y^2} \leq \frac{1}{t^2+1} \in L^1(\nu),$$

and $1/((t^2+y^2) \to 0$ as $y \to +\infty$. By the Lebesgue dominate convergence theorem,

$$\frac{\tilde{h}(iy)}{y} = \int_{\mathbb{R}} \frac{1}{t^2+y^2} d\nu(t) \to 0 \qquad \text{as} \quad y \to +\infty.$$

Suppose now in addition that $f$ has expansion as in (21). Then

$$c = \lim_{y \to +\infty} \frac{\operatorname{Im} f(iy)}{y} = \lim_{y \to +\infty} \frac{y + o(1)}{y} = 1.$$

So, by the definition of $c$, $\operatorname{Im} f(z) \geq \operatorname{Im} z$ for all $z \in \mathbb{H}$. Set $a_1 = \alpha + i\beta$, $z = x + iy \in \mathbb{H}$, $|z|$ large.

$$\operatorname{Im}\Big(\frac{a_1}{z}\Big) = \operatorname{Im}\Big(\frac{a_1 \bar{z}}{|z|^2}\Big) = \frac{1}{|z|^2}(\beta x - \alpha y).$$

Thus

$$0 \leq |z|(\operatorname{Im} f(z) - \operatorname{Im} z) = \frac{1}{|z|}(\beta x - \alpha y) + O\Big(\frac{1}{|z|}\Big).$$

So $\beta x - \alpha y \geq 0$ for $x + iy \in \mathbb{H}$. This implies that $\beta = 0$ and $\alpha \leq 0$. So $a_1 \in \mathbb{R}$ and $a_1 \leq 0$.

Case of equality: If $a_1 = 0$, then inductively, $a_2 = a_3 = \cdots = 0$.

Let $z = re^{i\varphi}$, $r > 0$, $\varphi \in (0, \pi)$. Suppose $a_1 = \cdot = a_{n-1} = 0$, inductively,

$$f(z) = z + \frac{a_n}{z^n} + \cdots.$$

So

$$0 \leq |z|^n (\operatorname{Im} f(z) - \operatorname{Im} z) = \operatorname{Im}(a_n e^{-in\varphi}) + O\Big(\frac{1}{|z|}\Big).$$

So $\operatorname{Im}(a_n e^{-in\varphi}) \geq 0$, $\varphi \in (0, \pi)$, equivalently, $\operatorname{Im}(a_n e^{i\alpha}) \geq 0$ for all $\alpha \in [0, 2\pi]$. This implies $a_n = 0$. □

**Theorem 7.9. (Integral representation)** *Let $\Omega \subseteq \mathbb{H}$ be a simply connected region such that $\mathbb{H} \setminus B(0, R) \subseteq \Omega$ for some $R > 0$. $f : \mathbb{H} \leftrightarrow \Omega$ be unique conformal map such that*

$$f(z) = z + \frac{a_1}{z} + \frac{a_2}{2} + \cdots \qquad \text{for } z \text{ near } \infty.$$

*Then there exists a unique finite Borel measure $\nu \geq 0$ on $\mathbb{R}$ with compact support such that*

$$f(z) = z + \int_{\mathbb{R}} \frac{1}{t - z} d\nu(t).$$

*Proof.* Uniqueness: follows form the uniqueness of Herglotz representation of $h = \operatorname{Im} f$.

Existence: Revisit proof of Herglotz representation. Let $\varphi : \mathbb{D} \cup \{1\} \leftrightarrow \mathbb{H} \cup \{\infty\}$ be conformal

Figure 25:

map defined by

$$z = \varphi(w) = i\frac{1 + w}{1 - w}, \qquad \text{and} \qquad \tilde{f} = f \circ \varphi.$$

By the Schwarz reflection principle, $\tilde{f}$ has a holomorphic extension across an open arc $\alpha \subseteq \partial\mathbb{D}$ with $1 \in \alpha$. $\tilde{f}(\alpha) \subseteq \hat{\mathbb{R}}$; $\tilde{f}(\alpha \setminus \{1\}) \subseteq \mathbb{R}$, $\operatorname{Im} \tilde{f}(\zeta) \equiv 0$ for $\zeta \in \alpha \setminus \{1\}$ (c.f. proof of Lemma 7.4), $\operatorname{Im} \tilde{f} > 0$ on $\mathbb{H}$. Let $\tilde{g} = -i\tilde{f}$. Then $\tilde{f} = i\tilde{g}$, $\operatorname{Re} \tilde{g} = \operatorname{Im} \tilde{f} > 0$, and $\operatorname{Re} \tilde{g}(\zeta) \equiv 0$ for $\zeta \in \alpha \setminus \{1\}$. So

$$\operatorname{Re} \tilde{g}(r\zeta) \to 0 \qquad \text{as} \quad r \to 1^-, \tag{22}$$

locally uniformly for $\zeta \in \alpha \setminus \{1\}$. In the Herglotz representation for $\tilde{g}$, the measure $\mu$ on $\partial\mathbb{D}$ can be obtained as $w^*$–limits of measure $\mu_r$ on $\partial\mathbb{D}$ as $r \to 1^-$, where

$$d\mu_r(\zeta) = \operatorname{Re} \tilde{g}(r\zeta) \frac{|d\zeta|}{2\pi}.$$

Then (22) implies that

$$\operatorname{supp}(\mu) \subseteq \partial\mathbb{D} \setminus (\alpha \setminus \{1\}) = \partial\mathbb{D} \setminus \alpha \cup \{1\}.$$

So

$$\tilde{f}(w) = b + i \int_{\partial\mathbb{D}} \frac{\zeta + w}{\zeta - w} d\mu(\zeta), \qquad \text{for some } b \in \mathbb{R}.$$

Going back to $\mathbb{H}$,

$$f(z) = az + b + \int_{\mathbb{R}} \Big( \frac{1 + t^2}{t - z} - t \Big) d\tau(t),$$

where $a = \mu(\{1\})$, $b \in \mathbb{R}$, and $\tau$ is finite measure with support in $\varphi(\partial\mathbb{D} \setminus \alpha) \Subset \mathbb{R}$. Let

$$d\nu(t) = (1 + t^2) d\tau(t), \qquad \tilde{b} = b + \int_{\mathbb{R}} t d\tau(t).$$

Then $\nu \geq 0$ is a finite measure with compact support, and

$$f(z) = az + \tilde{b} + \int_{\mathbb{R}} \frac{1}{t - z} d\nu(t) = az + \tilde{b} + O\Big(\frac{1}{z}\Big).$$

On the other hand, $f(z) = z + o(1)$, so $a = 1$, $\tilde{b} = 0$. $\square$

**Remark.** If $f : \mathbb{H} \leftrightarrow \Omega$ is as in Theorem 7.9, and $\operatorname{Im} f$ has a continuous extension to $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{R}$, then

$$f(z) = z + \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{t-z} \operatorname{Im} f(t) dt.$$

So

$$d\nu(t) = \frac{1}{\pi} \operatorname{Im} f(t) dt.$$

Note that if $f$ has a continuous extension to $\mathbb{R}$, then $\tilde{g}$ has a continuous extension to $\partial\mathbb{D} \setminus \{1\}$, and

$$d\mu(\zeta) = \frac{1}{2\pi} \operatorname{Re} \tilde{g}(\zeta) |d\zeta| \qquad \text{on} \quad \partial\mathbb{D} \setminus \{1\}.$$

Set $w = (z-i)/(z+i)$, $\zeta = (t-i)/(t+i)$. Then $d\zeta/dt = 2i/(t+i)^2$, $|d\zeta/dt| = 2/(1+t^2)$,

$$d\tau(t) = \frac{1}{2\pi} \operatorname{Re} \tilde{g}(\zeta) |d\zeta| = \frac{1}{2\pi} \operatorname{Im} f(t) \left| \frac{d\zeta}{dt} \right| dt = \frac{1}{\pi(1+t^2)} \operatorname{Im} f(t) dt.$$

Note that for $|z|$ large,

$$\begin{aligned} \int_{\mathbb{R}} \frac{1}{t-z} d\nu(t) &= -\frac{1}{z} \int_{\mathbb{R}} \frac{1}{1-t/z} d\nu(t) \\ &= -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \int_{\mathbb{R}} t^n d\nu(t). \qquad \text{(uniformly converges)} \end{aligned}$$

If $f(z) = z + \sum_{n=1}^{\infty} a_n/z^n$ is the Laurent expansion of $f$ near $\infty$, then

$$a_n = -\int_{\mathbb{R}} t^{n-1} d\nu(t) \le 0 \qquad \text{for} \quad n \in \mathbb{N},$$

if $a_1 = 0$, then $\nu(\mathbb{R}) = 0$, and $\nu = 0$. So $f(z) = z$.

The proof shows that $\operatorname{supp}(\nu) \Subset I$, if $I$ is an interval such that $f$ has a holomorphic extension to $\mathbb{R} \backslash I$ with $f(\mathbb{R} \backslash I) \subseteq \mathbb{R}$. In particular, if the Laurent expansion converges outside $\overline{B}(0,R)$, then $\operatorname{supp}(\nu) \subseteq [-R, R]$, and conversely, the integral representation shows that if $\operatorname{supp}(\nu) \subseteq [-R, R]$, then the Laurent expansion converges in $\mathbb{C} \setminus \overline{B}(0,R)$.

**Definition 7.10.** a) Let $K \subseteq \mathbb{C}$ be a set. Then $\operatorname{rad}(K) = \sup\{|z| : z \in K\}$.

b) Let $A$ be a set. $A \subseteq \mathbb{H}$ is called an $\mathbb{H}$-*hull* if $A$ is relatively closed in $\mathbb{H}$, i.e., $A = \overline{A} \cap \mathbb{H}$, and if $\Omega_A = \mathbb{H} \setminus A$ is a simply connected region, then there exists a unique conformal map $f_A : \mathbb{H} \leftrightarrow \Omega_A$ with holomorphic extension near $\infty$ of the form

$$f_A(z) = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots.$$

We call $\operatorname{hcap}(A) := -a_1 \ge 0$ the *half-plane capacity of* $A$.

c) $\mathcal{Q}$ = set of all $\mathbb{H}$-hulls.

**Lemma 7.11.** *Let $A$ be an $\mathbb{H}$-hull,*

$$f_A(z) = z + \int_{\mathbb{R}} \frac{1}{t-z} d\nu_A(t)$$

*be the integral representation as in 7.10. Then*

a) $\nu_A(\mathbb{R}) = \operatorname{hcap}(A)$,

b) $\operatorname{rad}(\operatorname{supp}(\nu_A)) \simeq \operatorname{rad}(A)$,

c) $\operatorname{hcap}(A) \lesssim \operatorname{rad}(A)^2$.

*Proof.* a) Suppose $f_A$ has the Laurent expansion $f_A(z) = z + a_1/z + \cdots$ near $\infty$, then

$$\operatorname{hcap}(A) = -a_1 = \int_{\mathbb{R}} d\nu_A(t) = \nu_A(\mathbb{R}).$$

b) We know that $R := \operatorname{rad}(\operatorname{supp}(\nu_A))$ is the smallest number such that the Laurent expansion of $f_A$ converges on $\mathbb{C} \setminus \overline{B}(0,R)$. Then by the Schwarz reflection principle, $f_A$ has a holomorphic extension to a conformal map on $\hat{\mathbb{C}} \setminus \overline{B}(0,R)$ into $\hat{\mathbb{D}}$. Define

$$h(w) := \frac{1}{R} f_A(Rw) = w + \frac{\tilde{a}_1}{w} + \cdots \qquad \text{for} \quad w \in \mathbb{D}^* := \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}.$$

Then $h \in \Sigma$ (c.f. Section 1), and so

$$\hat{\mathbb{C}} \setminus g(\mathbb{D}^*) \subseteq \overline{B}(0,2), \qquad \text{(c.f. Corollary 1.3)}$$

So $\frac{1}{R} A \subseteq \overline{B}(0,2)$, and so $A \subseteq \overline{B}(0,2R)$, i.e., $\operatorname{rad}(A) \leq 2R$.

Conversely, let $\tilde{R} = \operatorname{rad}(A)$. Then $g_A = f_A^{-1}$ has a conformal extension to $\hat{\mathbb{C}} \setminus \overline{B}(0,\tilde{R})$. Let

$$\tilde{h}(w) := \frac{1}{\tilde{R}} g_A(\tilde{R}w) = w + \frac{b_1}{w} + \cdots.$$

Then $\tilde{h} \in \Sigma$, and $\tilde{h}(\mathbb{D}^*) \supseteq \mathbb{C} \setminus \overline{B}(0,2)$, i.e.,

$$g_A(\mathbb{C} \setminus \overline{B}(0,\tilde{R})) \supseteq \mathbb{C} \setminus \overline{B}(0,2\tilde{R}).$$

So $f_A$ is holomorphic on $\mathbb{C} \setminus \overline{B}(0,2\tilde{R})$, i.e., $R \leq 2\tilde{R} = 2\operatorname{rad}(A)$. So $R \simeq \tilde{R}$.

c) Notation as in b). $f_A(z) = z + a_1/z + \cdots$,

$$h(w) = \frac{1}{R} f_A(Rw) = z + \frac{a_1}{R^2 z} + \cdots \in \Sigma.$$

By the Area Theorem 1.2, $|a_1/R^2| \leq 1$, and so

$$\operatorname{hcap}(A) = -a_1 \leq R^2 \lesssim \tilde{R}^2 = \operatorname{rad}(A)^2. \qquad \square$$

**Remark 7.12.** Let $\mathcal{A}$ be a family of $\mathbb{H}$-hulls, $\mathcal{F} = \{f_A : A \in \mathcal{A}\}$ be corresponding family of conformal maps $f_A : \mathbb{H} \leftrightarrow \mathbb{H} \setminus A$ with usual normalization $f_A(z) = z + o(1)$ near $\infty$. If $\operatorname{rad}(A)$ is uniformly bounded for $A \in \mathcal{A}$ (i.e., if $\{\operatorname{rad}(A) : A \in \mathcal{A}\}$ bounded), then one has good "a priori" control for the maps in $\mathcal{F}$. For example,

i) $f_A(z) = z + \int_{\mathbb{R}} \frac{1}{t-z} d\mu_A(t)$, where measures $\mu_A$ have uniformly bounded total mass with supports contained in a fixed interval (follows from Lemma 7.11).

ii) $\mathcal{F}$ is locally uniformly bounded, and in particular, a normal family. Actually, $\mathcal{F}$ is uniformly bounded on bounded subsets of $\mathbb{H}$. There exists $R > 0$ such that $f_A \in \mathcal{F}$ has extension to a conformal map on $\hat{\mathbb{C}} \setminus \overline{B}(0,R)$. Let $h_A(w) = \frac{1}{R} f_A(Rw)$, $w \in \mathbb{D}^*$. Then $h_A \in \Sigma$, and $h_A(\mathbb{D}^*) \supseteq \hat{\mathbb{C}} \setminus \overline{B}(0,2)$. So $f_A(B(0,R) \cap \mathbb{H}) \subseteq B(0,2R)$.

**7.13. Chordal Loewner chians (half-plane version of Loewner chains)**

Let $I = [0, b]$, $b \in (0, \infty]$. $\{\Omega_t\}_{t\in I}$ is a *(geometric) chordal Loewner chain* if

i) each $\Omega_t \subseteq \mathbb{H}$ is a simply connected region of the form $\Omega_t = \mathbb{H} \setminus A_t$, where $A_t$ is an $\mathbb{H}$-hull.

ii) $\Omega_0 = \mathbb{H}$ ($A_0 = \varnothing$).

iii) $\Omega_s \subsetneq \Omega_t$ for $s > t$, $s, t \in I$ (equivalently, $A_s \supsetneq A_t$).

iv) $\{\Omega_t\}_{t\in I}$ satisfies a continuity requirement (cf. Lemma 7.14).

If $f_t : \mathbb{H} \leftrightarrow \Omega_t$ be the unique conformal map such that

$$f_t(z) = z + \frac{a_1(t)}{z} + \frac{a_2(t)}{z^2} + \cdots, \qquad \text{near} \quad \infty,$$

then $\{f_t\}_{t\in I}$ is the corresponding *(analytic) chordal Loewner chain.* It is normalized if

$$f_t(z) = z - \frac{2t}{z} + \cdots, \qquad \text{near} \quad \infty \quad \text{for} \quad t \in I,$$

i.e., $a_1(t) = -2t$, $t \in I$.

**Lemma 7.14.** *Let $\{\Omega_t\}_{t\in I}$ be a chordal Loewner chain corresponding to analytic Loewner chain $\{f_t\}_{t\in I}$. Let $\{t_n\}$ be a sequence in $I$ with $t_n \to t_\infty$ as $n \to \infty$. Denote $\Omega_n = \Omega_{t_n}$, $f_n = f_{t_n}$, $\Omega_n = \mathbb{H} \setminus A_n$, and*

$$f_n(z) = z + \int_{\mathbb{R}} \frac{d\mu_n(u)}{u - z}, \qquad z \in \mathbb{H}.$$

*Then the following are equivalent:*

i) *$f_n \to f_\infty$ locally uniformly on $\mathbb{H}$.*

ii) *$\mu_n \xrightarrow{w^*} \mu_\infty$, i.e.,*

$$\int \varphi d\mu_n \to \int \varphi d\mu_\infty \qquad \textit{for all } \varphi \in C_c(\mathbb{R}) \quad (\textit{equivalently, for all } \varphi \in C(\mathbb{R})).$$

iii) $\operatorname{hcap}(A_n) \to \operatorname{hcap}(A_\infty)$.

iv) *$\Omega_n \to \Omega_\infty$ in the sense of kernel convergence with respect to $\infty$, where the kernel of $\{\Omega_n\}$ with respect to $\infty$, $\operatorname{Kern}_\infty(\{\Omega_n\})$ = the set of all points $w \in \mathbb{C}$ for which there exists an unbounded region $U$ with $w \in U$ and $U \subseteq \Omega_n$ for all large $n$.*

*Proof.* Let $T = \sum\{t_n : n \in \mathbb{N} \cup \{\infty\}\} \in I$. So $A_n \subseteq A_T$ and $\operatorname{rad}(A_n) \le \operatorname{rad}(A_T) < \infty$ for $n \in \mathbb{N} \cup \{\infty\}$. In particular, $f_n$, $n \in \cup\{\infty\}$, is uniformly bounded on bounded subsets of $\mathbb{H}$ and there exist $C_0 \ge 0$, $R_0 \ge 0$, such that

$$\mu_n(\mathbb{R}) \le C_0, \qquad \operatorname{supp}(\mu_n) \subseteq [-R_0, R_0] \qquad \text{for } n \in \mathbb{N} \cup \{\infty\}.$$

i) $\Longrightarrow$ ii).

I) If $\psi \in C_c(\mathbb{R}^2)$ is arbitrary, then

$$\int_{\mathbb{H}} f_n \psi dA \to \int_{\mathbb{H}} f_\infty \psi dA.$$

Suppose $\operatorname{supp}(\psi) \subseteq \overline{B}(0, R)$ and let $K_\delta = \{z \in \overline{B}(0, R) : \operatorname{Im} z \ge \delta\}$ for $\delta > 0$. Then

$$\begin{aligned}
\left|\int_{\mathbb{H}} (f_n - f_\infty)\psi\right| &\le \int_{\mathbb{H}\cap\overline{B}(0,R)} |\psi| \cdot |f_n - f_\infty| dA \\
&\le A(K_\delta)\|\psi\|_\infty \cdot \sup_{z\in K_\delta} |f_n(z) - f_\infty(z)| \\
&\quad + 4\delta R\|\psi\|_\infty \sup\{|f_n(z)| : n \in \mathbb{N} \cup \{\infty\}, z \in \overline{B}(0, R) \cap \mathbb{H}\} \\
&\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

if $\delta > 0$ is sufficiently small and $n$ is sufficiently large.

II) Let $P$ be an arbitrary polynomial (in $z$). Then

$$\int P d\mu_n \to \int P d\mu_\infty.$$

Pick $\chi \in C_c^\infty(\mathbb{C})$ such that $\chi|_{\overline{B}(0,R)} \equiv 1$, and $h = \chi \cdot P$. Then $h_{\overline{z}} = \chi_{\overline{z}} \cdot P \in C_c^\infty(\mathbb{C})$. Hence

$$h(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{h_{\overline{z}}(w)}{w-z} dA(w), \qquad z \in \mathbb{C}.$$

So

$$\begin{aligned}
\int_{\mathbb{R}} P d\mu_n &= \int_{\mathbb{R}} \chi P d\mu_n = \int_{\mathbb{R}} h d\mu_n = -\frac{1}{\pi} \int_{\mathbb{R}} \int_{\mathbb{C}} \frac{h_{\overline{z}}(w)}{w-u} dA(w) d\mu_n(u) \\
&= \frac{1}{\pi} \int_{\mathbb{C}} \left( \int_{\mathbb{R}} \frac{d\mu_n(u)}{u-w} \right) h_{\overline{z}}(w) dA(w) = \frac{1}{\pi} \int_{\mathbb{C}} (f_n(w) - w) h_{\overline{z}}(w) dA(w) \\
&\to \frac{1}{\pi} \int_{\mathbb{C}} (f_\infty(w) - w) h_{\overline{z}}(w) dA(w) = \int_{\mathbb{R}} P d\mu_\infty.
\end{aligned}$$

III) Let $\varphi \in C(\mathbb{R})$ be arbitrary. By the Weierstrass Approximation Theorem, there exists a polynomial such that $|P - \varphi| < \varepsilon$ on $[-R_0, R_0]$. Then

$$\left| \int \varphi d\mu_n - \int \varphi d\mu_\infty \right| \le \varepsilon \mu_n(\mathbb{R}) + \varepsilon \mu_\infty(\mathbb{R}) + \left| \int P d\mu_n - \int P d\mu_\infty \right| \le (2C_0 + 1)\varepsilon$$

for $n$ large.

ii) $\implies$ iii)

Suppose $\mu_n \xrightarrow{w^*} \mu_\infty$. Then

$$\text{hcap}(A_n) = \mu_n(\mathbb{R}) = \int_{\mathbb{R}} 1 d\mu_n \to \int_{\mathbb{R}} 1 d\mu_\infty = \mu_\infty(\mathbb{R}) = \text{hcap}(A_\infty).$$

iii) $\implies$ i)

Suppose $\text{hcap}(A_n) \to \text{hcap}(A_\infty)$. We want to show that $f_n \to f_\infty$ locally uniformly on $\mathbb{H}$. Equivalently, for all sequence $\{z_n\}$ in $\mathbb{H}$ with $z_n \to z_\infty \in \mathbb{H}$, we have $f_n(z_n) \to f_\infty(z_\infty)$.

**Spacial case I.** $t_\infty \le t_n$ for all $n \in \mathbb{N}$. Then $A_\infty \subseteq A_n$, equivalently, $\Omega_\infty \supseteq \Omega_n$. Let $\varphi_n := f_\infty^{-1} \circ f_n$, $n \in \mathbb{N}$, equivalently, $f_n = f_\infty \circ \varphi_n$. Then $\varphi_n(\mathbb{H}) \subseteq \mathbb{H}$, and $\varphi_n$ is conformal near $\infty$. Let $\varphi_n(\mathbb{H}) = \mathbb{H} \setminus B_n$, where $B_n$ is a $\mathbb{H}$-hull.

Let $a_n = \text{hcap}(A_n)$, $a_\infty = \text{hcap}(A_\infty)$. Then

$$f_n(z) = z + \frac{a_n}{z} + \cdots, \qquad f_\infty(z) = z + \frac{a_\infty}{z} + \cdots,$$

and

$$\varphi_n(z) = z + \frac{a_n - a_\infty}{z} + \cdots.$$

So

$$\text{hcap}(B_n) = a_n - a_\infty = \text{hcap}(A_n) - \text{hcap}(A_\infty) \to 0 \qquad \text{as } n \to \infty.$$

Write

$$\varphi_n(z) = z + \int_{\mathbb{R}} \frac{1}{u-z} d\nu_n(u),$$

where $\nu_n \geq 0$, $\operatorname{supp}(\nu_n) \Subset \mathbb{R}$. Then $\nu_n(\mathbb{R}) = \operatorname{hcap}(B_n) \to 0$. Since

$$|\varphi_n(z) - z| \leq \frac{\nu_n(\mathbb{R})}{\operatorname{Im} z} \qquad \text{for } z \in \mathbb{H},$$

we have $\varphi_n \to \mathrm{id}_{\mathbb{H}}$ locally uniformly on $\mathbb{H}$. If $z_n \in \mathbb{H} \to z_\infty \in \mathbb{H}$, then $\varphi_n(z_n) \to z_\infty$, and so $f_n(z_n) = f_\infty(\varphi_n(z_n)) \to f_\infty(z_\infty)$.

**Spacial case II.** $t_n \leq t_\infty$ for all $n \in \mathbb{N}$. In this case $A_n \subseteq A_\infty$, equivalently, $\Omega_n \supseteq \Omega_\infty$. Let $\varphi_n = f_n^{-1} \circ f_\infty$, equivalently, $f_n \circ \varphi_n = f_\infty$. Then $\varphi_n(\mathbb{H}) \subseteq \mathbb{H}$, and $\varphi_n$ is conformal near $\infty$. Let $\varphi_n(\mathbb{H}) = \mathbb{H} \setminus B_n$, where $B_n$ is a $\mathbb{H}$-hull. Similarly, we have

$$\operatorname{hcap}(B_n) = \operatorname{hcap}(A_\infty) - \operatorname{hcap}(A_n) \to 0 \implies \varphi_n \to \mathrm{id}_{\mathbb{H}}$$

locally uniformly on $\mathbb{H}$. If $z_n \in \mathbb{H} \to z_\infty \in \mathbb{H}$, then $\varphi_n(z_n) \to z_\infty$. From Remark 7.12, $\{f_n\}$ is a normal family. So $\{f_n\}$ is equicontinuous at $z_\infty$. We have

$$\begin{aligned} f_\infty(z_n) &= f_\infty(z_\infty) + o(1) \\ f_\infty(z_n) &= f_n(\varphi(z_n)) = f_n(z_\infty) + o(1) \\ f_n(z_n) &= f_n(z_\infty) + o(1). \end{aligned}$$

So

$$f_n(z_n) = f_\infty(z_\infty) + o(1).$$

Special case I + II imply general case.

i) $\implies$ iv)

Assume $f_n \to f_\infty$ locally uniformly on $\mathbb{H}$. We want to show that $\mathrm{Kern}_\infty := \mathrm{Kern}_\infty(\{\Omega_n\}) = \Omega_\infty$ (applied to all subsequences gives $\Omega_n \to \Omega_\infty$ with respect to $\infty$).

Note that $\operatorname{rad}(A_n) \leq \tilde{R} < \infty$ for $n \in \mathbb{N} \cup \{\infty\}$; so $U := \mathbb{H} \setminus \overline{B}(0, \tilde{R}) \subseteq \Omega_n$, $n \in \mathbb{N} \cup \{\infty\}$.

I. $\Omega_\infty = f_\infty(\mathbb{H}) \subseteq \mathrm{Kern}_\infty$.

Let $w \in \Omega_\infty$ be arbitrary. Then there exists $V \Subset \Omega_\infty$ open with $w \in V$ and $U \cap V \neq \varnothing$. It is enough to show that $V \subseteq \Omega_n$ for large $n$ ($\Rightarrow w \in \mathrm{Kern}_\infty$). If not, there exist $n_k \in \mathbb{N} \to \infty$ and $w_k \in V \setminus \Omega_n$ (without lose of generality $w_k \to w_\infty \in \overline{V} \subseteq \Omega_\infty$) such that $f_{n_k} - w_k$ zero free on $\mathbb{H}$. Note that $f_{n_k} - w_k \to f_\infty - w_\infty$ locally uniformly on $\mathbb{H}$. Since $w_\infty \in \Omega_\infty$, so $f_\infty - w_\infty$ is not zero free. So $f_\infty - w_\infty \equiv 0$ by Hurwitz, and $f_\infty \equiv w_\infty$, contradiction!

II. $\mathrm{Kern}_\infty \subseteq \Omega_\infty$.

Note that there exist $R_1, R_1' > 0$ and $C_1, C_1' > 0$ such that

(1) $|f_n(z) - z| \leq C_1$ for $z \in \mathbb{H} \setminus \overline{B}(0, R_1)$,

(2) $|f_n^{-1}(w) - w| \leq C_1'$ for $w \in \mathbb{H} \setminus \overline{B}(0, R_1')$.

Let $w_\infty \in \mathrm{Kern}_\infty$ be arbitrary. We want to show $w_\infty \in \Omega_\infty$, i.e., there exists $z_\infty \in \mathbb{H}$ such that $f_\infty(z_\infty) = w_\infty$. Since $w_\infty \in \mathrm{Kern}_\infty$, there exists a region $V \Subset \mathbb{H}$ with $V \cap U \neq \varnothing$, $w_\infty \in V$, and $V \Subset \Omega_n$ for large $n$ (wlog, for all $n$). Then $W = U \cup V \subseteq \Omega_n \subseteq \mathbb{H}$. Let $g_n = f_n^{-1}|_W$.

**Claim.** $\{g_n\}$ is locally uniformly bounded and hence a normal family.

Proof by contradiction. Suppose not. Then there exist $K \subseteq W$ compact and a sequence $\{w_n\}$ in $K$ such that $\{g_n(w_n)\}$ is unbounded. Without lose of generality, $w_n \to w \in K$, $g_n(w_n) \to \infty$. Then $w_n = f_n(g_n(w_n)) = g_n(w_n) + O(1)$ by (1) and $w_n \to w_\infty$. Contradiction!

Using claim and passing to a subsequence, we may assume $g_n \to g_\infty \in H(W)$ locally uniformly on $W$. $g_n(W) \subseteq \mathbb{H}$, so $g_\infty(W) \subseteq \mathbb{H} \cup \mathbb{R}$.

**Claim.** $g_\infty(W) \subseteq \mathbb{H}$.

Otherwise, $g_\infty \equiv$const. by open mapping theorem. But by (2), $|g_\infty(w) - w| \leq C_1'$ for $w \in \mathbb{H}$ with $|w|$ large. Contradiction!

Define $z_\infty = g_\infty(w_\infty) \in \mathbb{H}$. Then

$$f_\infty(z_\infty) = \lim_{n\to\infty} f_n(g_n(w_\infty)) = w_\infty$$

since $f_n \to f_\infty$ is locally uniform convergence.

vi) $\implies$ i)

Assume $\Omega_n \to \Omega_\infty$. We want to show that $f_n \to f_\infty$ locally uniformly on $\mathbb{H}$. Since $\{f_n\}$ is a normal family, it suffices to show every subsequence $\{\tilde{f}_n\}$ of $\{f_n\}$ has a subsequence that converges to $f_\infty$ locally uniformly on $\mathbb{H}$. Write

$$\tilde{f}_n(z) = z + \int_{\mathbb{R}} \frac{d\tilde{\mu}_n(u)}{u - z},$$

where $\mathrm{supp}(\tilde{\mu}_n) \subseteq [-R_0, R_0]$, $\tilde{\mu}_n(\mathbb{R}) \leq C_0$. Passing to a subsequence, wlog, $\tilde{\mu}_n \xrightarrow{w^*} \tilde{\mu}_\infty$, where $\tilde{\mu}_\infty \geq 0$ is a measure supported on $[-R_0, R_0]$. Then

$$\int_{\mathbb{R}} \varphi d\tilde{\mu}_n \longrightarrow \int_{\mathbb{R}} \varphi d\tilde{\mu}_\infty \qquad \text{for all } \varphi \in C(\mathbb{R}).$$

So

$$\tilde{f}_n(z) = z + \int_{\mathbb{R}} \frac{d\tilde{\mu}_n(u)}{u - z} \longrightarrow \tilde{f}_\infty(z) = z + \int_{\mathbb{R}} \frac{d\tilde{\mu}_\infty(u)}{u - z}$$

pointwise for all $z \in \mathbb{H}$. Since $\{\tilde{f}_n\}$ is a normal family, $\tilde{f}_n \to \tilde{f}_\infty$ is locally uniformly on $\mathbb{H}$.

$\tilde{f}_\infty$ is a conformal map, $\tilde{f}_\infty(z) = z + o(1)$ near $\infty$, $\tilde{f}_\infty(\mathbb{H}) = \mathbb{H} \setminus \tilde{A}_\infty$, where $\tilde{A}_\infty$ is a $\mathbb{H}$-hull. By implication i) $\implies$ iv), we have

$$\tilde{\Omega}_\infty = \mathrm{Kern}_\infty(\{\tilde{\Omega}_n\}) = \Omega_\infty.$$

So both $f_\infty, \tilde{f}_\infty : \mathbb{H} \leftrightarrow \Omega_\infty = \tilde{\Omega}_\infty$ are conformal maps. Since

$$f_\infty(z) = z + o(1), \qquad \tilde{f}_\infty(z) = z + o(1), \qquad \text{near} \quad \infty,$$

by uniqueness (Corollary 7.5), $\tilde{f}_\infty = f_\infty$. So $\tilde{f}_n \to f_\infty$ locally uniformly on $\mathbb{H}$. □

**Lemma 7.15.** *Let $A, B$ be $\mathbb{H}$-hulls. Then*

i) $\mathrm{hcap}(A) \geq 0$ *with equality if and only if* $A = \varnothing$.

ii) $\mathrm{hcap}(x + A) = \mathrm{hcap}(A)$, $x \in \mathbb{R}$.

iii) $\mathrm{hcap}(\lambda A) = \lambda^2 \,\mathrm{hcap}(A)$, $\lambda > 0$.

iv) *Suppose* $A \subseteq B$. *Then* $\mathrm{hcap}(A) \leq \mathrm{hcap}(B)$ *with equality if and only if* $A = B$.

*Proof.* Let $f_A : \mathbb{H} \leftrightarrow \mathbb{H} \setminus A$ be conformal, with

$$f_A(z) = z + \frac{a_1}{z} + \cdots = z + \int_{\mathbb{R}} \frac{d\mu_A(u)}{u - z} \qquad \text{near } \infty.$$

Then $\operatorname{hcap}(A) = -a_1 = \mu_A(\mathbb{R})$.

i) So $\operatorname{hcap}(A) \geq 0$ with equality if and only if $\mu_A \equiv 0$ if and only if $f_A(z) \equiv z$ if and only if $\mathbb{H} \setminus A = \mathbb{H}$ if and only if $A = \varnothing$.

ii) Let $x \in \mathbb{R}$.

$$f_{x+A}(z) = x + f_A(z-x) = z + \frac{a_1}{z-x} + \cdots = z + \frac{a_1}{z} + \cdots .$$

So $\operatorname{hcap}(x+A) = \operatorname{hcap}(A)$.

iii) Let $\lambda > 0$.

$$f_{\lambda A}(z) = \lambda f_A(z/\lambda) = z + \frac{a_1 \lambda}{z/\lambda} + \cdots = z + \frac{\lambda^2 a_1}{z} + \cdots .$$

So $\operatorname{hcap}(\lambda A) = \lambda^2 \operatorname{hcap}(A)$.

iv) Let $\varphi = f_A^{-1} \circ f_B : \mathbb{H} \leftrightarrow \mathbb{H} \setminus C$. Then $\operatorname{hcap}(C) = \operatorname{hcap}(B) - \operatorname{hcap}(A) \geq 0$ with equality if and only if $C = \varnothing$ if and only if $\varphi = \mathrm{id}_{\mathbb{H}}$ if and only if $f_A = f_B$ if and only if $A = B$. □

**Remark 7.16.** Let $\{\Omega_t\}_{t \in I}$ be a chordal Loewner chain, $\Omega_t = \mathbb{H} \setminus A_t$, $A_t \Subset \mathbb{H}$ be $\mathbb{H}$-hull, $A_t \subsetneq A_s$ if $t < s$ and $A_0 = \varnothing$. The map $t \to \operatorname{hcap}(A_t)$ is continuous (Lemma 7.14) and strictly increasing (Lemma 7.15). So $t \to \operatorname{hcap}(A_t)$ is a homeomorphism of $I = [0,b]$ onto its image $J = [0,b']$. By reparametrizing $t$, we may assume that $\operatorname{hcap}(A_t) = 2t$ for $t \in I$. Then

$$f_t(z) = z - \frac{2t}{z} + \cdots \qquad \text{near} \quad \infty,$$

and $\{f_t\}$ is normalized. So, without lose of generality, one can assume that a chordal Loewner chain is normalized.

### 7.17. The associated semi-group

Let $\{f_t\}_{t \in I}$ be a chordal Loewner chain, $f_t : \mathbb{H} \leftrightarrow \Omega_t = \mathbb{H} \setminus A_t$. For $0 \leq t \leq s$, let $\varphi_{s,t} = f_t^{-1} \circ f_s$, or equivalently, $f_s = f_t \circ \varphi_{s,t}$. Then $\varphi_{s,t}$ satisfies the following semigroup property

$$\varphi_{t,u} \circ \varphi_{s,t} = \varphi_{s,u}, \quad 0 \leq u \leq t \leq s, \qquad \text{and} \qquad \varphi_{t,t} = \mathrm{id}_{\mathbb{H}} .$$

**Lemma 7.18.** *Let $\{f_t\}_{t \in I}$ be a normalized chordal Loewner chain with associated semigroup $\varphi_{s,t}$. Then for $t, s \in I$, $t \leq s$, $\varphi_{s,t}$ is a conformal map $\mathbb{H} \leftrightarrow \mathbb{H} \setminus B_{s,t}$, where $B_{s,t}$ is a $\mathbb{H}$-hull, and*

$$\varphi_{s,t}(z) = z - \frac{2(s-t)}{z} + \cdots \qquad \textit{near} \quad \infty.$$

*There exists a measure $\mu_{s,t} \geq 0$, $\operatorname{supp}(\mu_{s,t}) \Subset \mathbb{R}$ such that*

$$\varphi_{s,t}(z) = z + \int_{\mathbb{R}} \frac{d\nu_{s,t}(u)}{u - z}, \qquad z \in \mathbb{H}.$$

$\nu_{s,t}(\mathbb{R}) = 2(s-t)$. *Moreover, if $t \leq s \leq T$, then $\operatorname{rad}(B_{s,t}) \leq C_0$ (and so $\operatorname{supp}(\nu_{s,t})$ is uniformly bounded).*

*Proof.* Clear that $\varphi_{s,t} = f_t^{-1} \circ f_s$ has a conformal extension near $\infty$ that maps real axis near $\infty$ into itself. So $\varphi_{s,t}$ is conformal map of $\mathbb{H}$ onto $\mathbb{H}\backslash$compact set, i.e., $\varphi_{s,t}(\mathbb{H}) = \mathbb{H} \setminus B_{s,t}$, where $B_{s,t}$ is a $\mathbb{H}$-hull.

$$\operatorname{hcap}(B_{s,t}) = \operatorname{hcap}(A_s) - \operatorname{hcap}(A_t) = 2(s-t).$$

So

$$\varphi_{s,t}(z) = z - \frac{2(s-t)}{z} + \cdots \qquad \text{near} \quad \infty,$$

and

$$\varphi_{s,t}(z) = z + \int_{\mathbb{R}} \frac{d\nu_{s,t}(u)}{u-z}, \qquad z \in \mathbb{H}. \qquad\qquad \text{(Theorem 7.9)}$$

We know $\nu_{s,t} \geq 0$, $\operatorname{supp}(\nu_{s,t}) \Subset \mathbb{R}$, and $\nu_{s,t}(\mathbb{R}) = \operatorname{hcap}(B_{s,t}) = 2(s-t)$. Finally, $\operatorname{rad}(B_{s,t}) \leq 2\operatorname{rad}(A_s) \leq C_0$ for $t, s \leq T$. ☐

**Lemma 7.19.** *Let $\{f_t\}_{t\in I}$ be a normalized chordal Loewner chain. $\varphi_{s,t} = f_t^{-1} \circ f_s$, $t \leq s$, $s, t \in I$. Then for fixed $z \in \mathbb{H}$,*

i) $|\varphi_{s,t}(z) - z| \leq \dfrac{2(s-t)}{\operatorname{Im} z}$.

ii) $|f_t(z) - f_s(z)| \leq \dfrac{2(s-t)}{(\operatorname{Im} z)^3}[2t + (\operatorname{Im} z)^2]$.

iii) $|\varphi_{s,t}(z) - \varphi_{s,u}(z)| \leq \dfrac{2(t-u)}{\operatorname{Im} z}$, *for $u \leq t \leq s$, $u, t, s \in I$.*

iv) $|\varphi_{s,u}(z) - \varphi_{t,u}(z)| \leq \dfrac{2(s-t)}{(\operatorname{Im} z)^3}[2t + (\operatorname{Im} z)^2]$, *for $u \leq t \leq s$, $u, t, s \in I$.*

*So the maps $(z,t) \mapsto f_t(z)$, $(z,t) \mapsto \varphi_{s,t}(z)$, $(z,t) \mapsto \varphi_{t,u}(z)$ belong to $HL(\mathbb{H} \times I)$, $HL(\mathbb{H} \times [0,s])$, $HL(\mathbb{H} \times [u,b])$, respectively, where $I = [0,b]$.*

*Proof.* Recall

$$f_t(z) = z + \int_{\mathbb{R}} \frac{d\mu_t(u)}{u-z}, \qquad \mu_t(\mathbb{R}) = 2t,$$

$$\varphi_{s,t}(z) = z + \int_{\mathbb{R}} \frac{d\nu_{s,t}(u)}{u-z}, \qquad \nu_{s,t}(\mathbb{R}) = 2(s-t).$$

By Julia's Lemma on integral representation, $\operatorname{Im} \varphi_{s,t}(z) \geq \operatorname{Im} z$.

i) $|\varphi_{s,t}(z) - z| \leq \displaystyle\int_{\mathbb{R}} \frac{d\nu_{s,t}(u)}{|u-z|} \leq \frac{\nu_{s,t}(\mathbb{R})}{\operatorname{Im} z} = \frac{2(s-t)}{\operatorname{Im} z}$.

ii) $f_t'(z) = 1 - \displaystyle\int_{\mathbb{R}} \frac{d\mu_t(u)}{(u-z)^2}$, $|f_t'(z)| \leq 1 + \dfrac{2t}{(\operatorname{Im} z)^2}$.

$$\begin{aligned} |f_t(z) - f_s(z)| &\leq |f_t(z) - f_t(\varphi_{s,t}(z))| \\ &\leq |z - \varphi_{s,t}(z)| \cdot \left(1 + \frac{2t}{(\operatorname{Im} z)^2}\right) \leq \frac{2(s-t)}{(\operatorname{Im} z)^3}[2t + (\operatorname{Im} z)^2]. \end{aligned}$$

iii) $\varphi_{t,u} \circ \varphi_{s,t} = \varphi_{s,u}$. So

$$|\varphi_{s,t}(z) - \varphi_{s,u}(z)| = |\varphi_{s,t}(z) - \varphi_{t,u}(\varphi_{s,t}(z))| \overset{\text{i)}}{\leq} \frac{2(t-u)}{\operatorname{Im} \varphi_{s,t}(z)} \leq \frac{2(t-u)}{\operatorname{Im} z}.$$

iv) $|\varphi'_{t,u}(z)| = \left|1 - \int_{\mathbb{R}} \frac{d\nu_{t,u}(x)}{(x-z)^2}\right| \leq 1 + \frac{2(t-u)}{(\operatorname{Im} z)^2} \leq 1 + \frac{2t}{(\operatorname{Im} z)^2}$. So

$$\begin{aligned}|\varphi_{s,u}(z) - \varphi_{t,u}(z)| &= |\varphi_{t,u}(\varphi_{s,t}(z)) - \varphi_{t,u}(z)| \\ &\leq |\varphi_{s,t}(z) - z| \cdot \left(1 + \frac{2t}{(\operatorname{Im} z)^2}\right) \leq \frac{2(s-t)}{(\operatorname{Im} z)^3}[2t - (\operatorname{Im} z)^2]. \qquad \square\end{aligned}$$

**Corollary 7.20.** *Let $\{f_t\}_{t\in I}$ be a normalized chordal Loewner chain, $\varphi_{s,t} = f_t^{-1} \circ f_s$, $t \leq s$, $s, t \in I$. Denote $f(z,t) = f_t(z)$. Then there exists a set $E \subseteq I$ with $|E| = 0$ such that*

i) *$f$ is differentiable at each point $(z,t) \in \mathbb{H} \times I \setminus E$, i.e.,*

$$f(z',t') = f(z,t) + \frac{\partial f}{\partial z}(z,t)(z'-z) + \frac{\partial f}{\partial t}(z,t)(t'-t) + o(|t'-t| + |z'-z|) \quad \text{near } (z,t).$$

*In particular, $\partial f(z,t)/\partial t$ exists for all $(z,t) \in \mathbb{H} \times I \setminus E$.*

ii) $V(z,t) = \lim_{\varepsilon\to 0^+} \frac{\varphi_{t,t-\varepsilon}(z) - z}{\varepsilon}$ *exists for all $(z,t) \in \mathbb{H} \times I \setminus E$, and*

$$\frac{\partial f}{\partial t}(z,t) = V(z,t) \cdot \frac{\partial f}{\partial z}(z,t).$$

*Proof.* i) follows from Lemma 7.19 and Proposition 4.12.

ii) Let $(z,s) \in \mathbb{H} \times I \setminus E$, $t \leq s$, $t$ near $s$. $f_t \circ \varphi_{s,t} = f_s$, $z' = \varphi_{s,t}(z)$.

$$|z' - z| = |\varphi_{s,t}(z) - z| \leq C|s-t|, \qquad \text{(Lemma 7.19).}$$

$$\begin{aligned}0 &= f_t(\varphi_{s,t}(z)) - f_s(z) = f(z',t) - f(z,s) \\ &= \frac{\partial f}{\partial z}(z,s)(z'-z) + \frac{\partial f}{\partial t}(z,s)(t-s) + o(|t-s| + |z'-z|) \\ &= \frac{\partial f}{\partial z}(z,s)(z'-z) + \frac{\partial f}{\partial t}(z,s)(t-s) + o(|t-s|)\end{aligned}$$

Note that $\partial f(z,s)/\partial z \neq 0$. So

$$V(z,s) = \lim_{t\to s^-} \frac{\varphi_{s,t}(z) - z}{s-t} = \lim_{t\to s^-} \frac{z'-z}{s-t} = \lim_{t\to s^-} \frac{\dot{f}(z,s)}{f'(z,s)} + o(1) = \frac{\dot{f}(z,s)}{f'(z,s)}. \qquad \square$$

**Theorem 7.21. (Loewner-Kufarev equation for chordal case)** *Let $\{f_t\}_{t\in I}$ be a normalized chordal Loewner chain, $\varphi_{s,t} = f_t^{-1} \circ f_s$, $t \leq s$, $s,t \in I$. Denote $f(z,t) = f_t(z)$. Then there eixsts $E \subseteq I$ with $|E| = 0$ such that*

(a) $$V(z,t) = \lim_{\varepsilon\to 0^+} \frac{\varphi_{t,t-\varepsilon}(z) - z}{\varepsilon}$$

*exists for all $(z,t) \in \mathbb{H} \times I \setminus E$.*

(b) *$\partial f(z,t)/\partial t$ exists for all $z \in \mathbb{H}$, $t \in I \setminus E$, and*

$$\frac{\partial f}{\partial t}(z,t) = V(z,t) \frac{\partial f}{\partial t}(z,t). \qquad \text{(Loewner-Kufarev equation)}$$

*Moreover, $V(z,t)$ has the following properties:*

i) *$V(\cdot,t)$ is holomorphic on $\mathbb{H}$ for each $t \in I \setminus E$,*

ii) *$V$ is measurable on $\mathbb{H}\times I$,*

iii) *for each $t\in I\setminus E$, there exists a probability measure $\nu_t$ on $\mathbb{R}$, $\operatorname{supp}(\nu_t)\Subset\mathbb{R}$ such that*

$$V(z,t)=2\int_{\mathbb{R}}\frac{d\nu_t(u)}{u-z},\qquad t\in I\setminus E,\ z\in\mathbb{H}.$$

*Proof.* We know that there exists $E\subseteq I$, $|E|=0$, such that

$$V(z,t):=\lim_{\varepsilon\to0^+}\frac{\varphi_{t,t-\varepsilon}(z)-z}{\varepsilon}$$

exists for all $z\in\mathbb{H}$, $t\in I\setminus E$, $\partial f(z,t)/\partial t$ exists for all $z\in\mathbb{H}$, $t\in I\setminus E$, and

$$\frac{\partial f}{\partial t}(z,t)=V(z,t)\frac{\partial f}{\partial z}(z,t).$$

We know $\partial f(z,t)/\partial z\neq0$, $\partial f(\cdot,t)/\partial t\in H(\mathbb{H})$ for $t\in I\setminus E$ (Proposition 4.12). So

$$V(\cdot,t)=\frac{\dot f(\cdot,t)}{f'(\cdot,t)}\in H(\mathbb{H})\qquad\text{for } t\in I\setminus E,$$

and $V$ is measurable on $\mathbb{H}\times I$.

$$\frac{\varphi_{t,t-\varepsilon}(z)-z}{\varepsilon}=\frac{1}{\varepsilon}\int_{\mathbb{R}}\frac{d\nu_{t,t-\varepsilon}(u)}{u-z}.$$

Here $\nu_{t,t-\varepsilon}(\mathbb{R})=2\varepsilon$, $\operatorname{supp}(\nu_{t,t-\varepsilon})\Subset\mathbb{R}$. Actually, the supports of $\nu_{t,t-\varepsilon}$ are uniformly bounded for $\varepsilon>0$, $t$ fixed (Lemma 7.18), say $\operatorname{supp}(\nu_{t,t-\varepsilon})\subseteq[-R_0,R_0]$. Let

$$\tau_\varepsilon:=\frac{1}{2\varepsilon}\nu_{t,t-\varepsilon}.$$

Then $\tau_\varepsilon$ subconverges to a probability measure $\nu_t$ on $[-R_0,R_0]$ as $\varepsilon\to0$ with respect to $w^*$-convergence. So

$$\begin{aligned}V(z,t)&:=\lim_{\varepsilon\to0^+}\frac{\varphi_{t,t-\varepsilon}(z)-z}{\varepsilon}\\&=\lim_{\varepsilon\to0^+}2\int_{[-R_0,R_0]}\frac{1}{u-z}d\tau_\varepsilon=2\int_{[-R_0,R_0]}\frac{d\nu_t(u)}{u-z},\qquad z\in\mathbb{H},\ t\in I\setminus E.\end{aligned}$$

□

**Remark 7.22.** The following are equivalent:

i)
$$V(z)=\int_{\mathbb{R}}\frac{d\nu(u)}{u-z}\qquad\text{for } z\in\mathbb{H}$$

where $\nu\geq0$, $\nu(\mathbb{R})=1$, and $\operatorname{supp}(\nu)\geq0$.

ii) $V$ is holomorphic on $\mathbb{H}$, $\operatorname{Im}V(z)\geq0$ for $z\in\mathbb{H}$, $V$ has a holomorphic extension near $\infty$ such that

$$V(z)=-\frac{1}{z}+\cdots\qquad\text{near}\quad\infty,$$

and $\operatorname{Im}f(x)=0$ for $x\in\mathbb{R}$, $|x|$ large.

*Proof.* i) $\Longrightarrow$ ii) Let $z = x + iy$.

$$\operatorname{Im}\Big(\frac{1}{u-z}\Big) = \frac{y}{(u-x)^2+y^2} > 0, \qquad \text{for } z \in \mathbb{H}.$$

ii) $\Longrightarrow$ i) Follows as in the proof of Theorem 7.9 from Herglotz representation. Note that if $\operatorname{Im} V$ has a continuous extension to $\mathbb{R}$, then

$$d\nu(u) = \frac{1}{\pi}\operatorname{Im} V(u)du. \qquad \square$$

**Example 7.23.** $\Omega_s = \mathbb{H} \setminus [0, is]$

Figure 26:

$$z = i\sqrt{-w^2-s^2} = \sqrt{w^2+s^2} = w\sqrt{1+\frac{s^2}{w^2}} = w + \frac{s^2}{2w} + \cdots \qquad \text{near } \infty.$$

Let $2t = s^2/2$, $s^2 = 4t$. Then $z = \sqrt{w^2+4t}$ or $z^2 = w^2 + 4t$ or $w = f_t(z) = \sqrt{z^2-4t}$, which is the normalized Loewner chain.

$$\dot{f}_t(z) = -\frac{2}{\sqrt{z^2-4t}}, \qquad f_t'(z) = \frac{z}{\sqrt{z^2-4t}},$$

$$V(z,t) = \frac{\dot{f}_t(z)}{f_t'(z)} = -\frac{2}{z} = 2\int_{\mathbb{R}} \frac{d\delta_0(u)}{u-z}.$$

So $\nu_t = \delta_0$ for all $t \geq 0$.

$$f_t(z) = z + \frac{1}{\pi}\int_{\mathbb{R}} \frac{\operatorname{Im} f_t(z)}{u-z}du,$$

$$\operatorname{Im} f_t(z) = \begin{cases} \sqrt{4t-u^2} & \text{for } \quad u \in [-2\sqrt{t}, 2\sqrt{t}] \\ 0 & \qquad\quad \text{elsewhere} \end{cases}$$

$$f_t(z) = z + \int_{\mathbb{R}} \frac{d\mu_t(u)}{u-z},$$

where

$$d\mu_t(u) = \frac{1}{\pi}\sqrt{4t-u^2}\chi_{[-2\sqrt{t},2\sqrt{t}]}(u)du, \qquad t \geq 0. \qquad \text{(semi-circle law)}$$

$$\mu_t(\mathbb{R}) = \mu_t(2t) = \frac{1}{\pi}\frac{\pi}{2}(2\sqrt{t})^2 = 2t.$$

**Example 7.24.** $\Omega_s = \mathbb{H} \setminus \overline{B}(0, s)$. Using Joukowski function $v = u + 1/u$.

$$z = s\Big(\frac{w}{s} + \frac{s}{w}\Big) = w + \frac{s^2}{w} \overset{2t=s^2}{=} w + \frac{2t}{w}.$$

$$w^2 - zw + 2t = 0, \quad w = \frac{z}{2} + \sqrt{\frac{z^2}{4} - 2t} = \frac{1}{2}\left(z + \sqrt{z^2-8t}\right).$$

So

$$f_t(z) = \frac{1}{2}\left(z + \sqrt{z^2-8t}\right), \qquad \text{(normalized Loewner chain)}$$

$$\dot{f}_t(z) = -\frac{2}{\sqrt{z^2-8t}}, \qquad f_t'(z) = \frac{1}{2}\Big(1+\frac{z}{\sqrt{z^2-8t}}\Big).$$

$$V(z,t) = \frac{\dot{f}_t(z)}{f_t'(z)} = \cdots = -\frac{1}{2t}\big(z-\sqrt{z^2-8t}\big),$$

$$\operatorname{Im} V(u,t) = \begin{cases} \frac{1}{2t}\sqrt{8t-u^2} & \text{for} \quad u\in[-\sqrt{8t},\sqrt{8t}] \\ 0 & \quad\quad\;\; \text{elsewhere} \end{cases}.$$

$$V(z,t) = 2\int_{\mathbb{R}} \frac{d\nu_t(u)}{u-z}, \qquad d\nu_t(u) = \frac{1}{4\pi t}\sqrt{8t-u^2}\,\chi_{[-\sqrt{8t},\sqrt{8t}]}(u)du,$$

$$\operatorname{Im} f_t(u) = \begin{cases} \frac{1}{2}\sqrt{8t-u^2} & \text{for} \quad u\in[-\sqrt{8t},\sqrt{8t}] \\ 0 & \quad\quad\;\; \text{elsewhere} \end{cases}$$

$$f_t(z) = z + \int_{\mathbb{R}} \frac{d\mu_t(u)}{u-z}, \qquad d\mu_t(u) = \frac{1}{2\pi}\sqrt{8t-u^2}\,\chi_{[-\sqrt{8t},\sqrt{8t}]}(u)du,$$

$$\mu_t(\mathbb{R}) = 2t, \qquad \frac{1}{2t}\mu_t = \nu_t.$$

# 8 Basic probabilistic concepts

### 8.1. Probability space

Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a probability space, where
$\Omega$ is a sample space, the space of outcomes. $\omega \in \Omega$ is a elementary outcome or event.
$\mathscr{A}$ is a $\sigma$-algebra or "$\sigma$-field". $A \in \mathscr{A}$ is an event.
$\mathbb{P}$ is a probability measure defined on $\mathscr{A}$, $\mathbb{P} \geq 0$ and $\mathbb{P}(\Omega) = 1$.

**Example 8.2.** Let $\Omega = \{1,2,3,4,5,6\}$, $\mathscr{A} = \wp(\Omega)$, $\mathbb{P} = 1/6 \cdot$ counting measure. Pick $\omega \in \Omega$ "at random" = roll a dice.

### 8.3. random variables

A measurable map $X : \Omega \to \mathbb{R}$ is called a *random variable* (i.e., $X^{-1}(B) \in \mathscr{A}$ for each Borel set $B \subseteq \mathbb{R}$).

$$\mathbb{E}[X] := \int_\Omega X(\omega)\, d\,\mathbb{P}(\omega)$$

is called the *expectation* or *mean* of $X$.

$$\operatorname{Var}(X) = \mathbb{E}[(X-\mathbb{E}[X])^2] = \int_\Omega (X-\mathbb{E}[X])^2 d\,\mathbb{P} = \mathbb{E}[X^2] - [X]^2$$

is called the *variance* of $X$.

**Lemma 8.4. (Borel-Cantelli-I)** *Let $A_n$, $n \in \mathbb{N}$, be events. If $\sum_{n=1}^\infty \mathbb{P}(A_n) < \infty$, then*

$$\mathbb{P}(A_{n,\text{i.o.}}) = 0,$$

*where $A_{n,\text{i.o.}}$ means that events in $\{A_n\}$ infinitely often occur. That is,*

$$A_{n,\text{i.o.}} = \{\omega\in\Omega : \omega\in A_n \textit{ for infinitely many } n\} = \bigcap_{k=1}^\infty \bigcup_{n\geq k} A_n.$$

*Proof.* $$\mathbb{P}(A_{n,\text{i.o.}}) = \lim_{k\to\infty} \mathbb{P}(\bigcup_{n\geq k} A_n) \leq \limsup_{k\to\infty} \sum_{n=k}^{\infty} \mathbb{P}(A_n) = 0.$$ □

**Lemma 8.5. (Chebyshev Inequality)** *If $X \geq 0$, then*

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[X]}{a}, \qquad a > 0.$$

*Proof.* $$\mathbb{P}(X \geq a) = \int_\Omega \chi_{X\geq a}(\omega)\, d\,\mathbb{P}(\omega) \leq \int_\Omega \frac{1}{a} X\, d\,\mathbb{P} = \frac{\mathbb{E}[X]}{a}.$$ □

**8.6. The distribution of a random variable**

Let $X : \Omega \to \mathbb{R}^n$ be random variable. The *distribution* or *law* of $X$ is the push-forward measure $\mathbb{P}_X := X_*\mathbb{P}$ on $\mathbb{R}^n$, i.e.,

$$\mathbb{P}_X(B) = \mathbb{P}(X^{-1}(B)) \qquad \text{for each Borel set } B \subseteq \mathbb{R}^n.$$

We have

$$\mathbb{E}[X] = \int_{\mathbb{R}} x\, d\,\mathbb{P}_X(x).$$

The *characteristic function* of $X : \Omega \to \mathbb{R}^n$ is defined by

$$f(u) := \mathbb{E}[e^{iu\cdot X}] \qquad \text{for} \quad u \in \mathbb{R}^n.$$

or

$$f(u) = \int_\Omega e^{iu\cdot X(\omega)} d\,\mathbb{P}(\omega) = \int_{\mathbb{R}^n} e^{iu\cdot v} d\,\mathbb{P}_X(v)$$
$$= \text{the Fourier transform of its distribution.}$$

Let $X_1, \ldots, X_n : \Omega \to \mathbb{R}$ be random variables, and let $X = (X_1, \ldots, X_n) : \Omega \to \mathbb{R}^n$. Then the *joint law* of $X_1, \ldots, X_n$ is defined to be the law of $X$.

**8.7. Independence**

Let $A, B \in \mathscr{A}$ be events. $A$ and $B$ are *independent* if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B).$$

Denote $A^c := \Omega \setminus A$. Then if $A, B$ are independent, then $A^c, B$ are independent. In fact

$$\mathbb{P}(A^c \cap B) = \mathbb{P}(B) - \mathbb{P}(A \cap B) = \mathbb{P}(B)(1 - \mathbb{P}(A)) = \mathbb{P}(A^c)\mathbb{P}(B).$$

If $\mathscr{F}_1, \ldots, \mathscr{F}_n \subseteq \mathscr{A}$ are $\sigma$-algebras. $\mathscr{F}_1, \ldots, \mathscr{F}_n$ are independent if

$$\mathbb{P}(A_1 \cap \cdots \cap A_n) = \mathbb{P}(A_1) \cdots \mathbb{P}(A_n)$$

whenever $A_1 \in \mathscr{F}_1, \ldots, A_n \in \mathscr{F}_n$.

$A, B$ are independent iff the $\sigma$-algebras generated by $A$ and by $B$ are independent.

Let $X_1, \ldots, X_n$ are random variables. They are independent if the $\sigma$-algebras $\sigma(X_1), \ldots, \sigma(X_n)$ generated by them are independent, where for a random variable $X$,

$$\sigma(X) = \{X^{-1}(B) : B \subseteq R^n \text{ Borel}\}.$$

If $X_1, \ldots, X_n$ are independent, and $f_1, \ldots, f_n : \mathbb{R} \to \mathbb{R}$ are Borel, then $f_1(X_1), \ldots, F_n(X_n)$ are independent. Note that $\sigma(f(X)) \subseteq \sigma(X)$.

**Theorem 8.8.** *Let $X_1, \dots, X_n : \Omega \to \mathbb{R}$ be random variables, and let $X = (X_1, \dots, X_n) : \Omega \to \mathbb{R}^n$. then TFAE:*

(i) *$X_1, \dots, X_n$ are independent,*

(ii) *$\mathbb{P}(X_1 \in B_1, \dots, X_n \in B_n) = \mathbb{P}(X_1 \in B_1) \cdots \mathbb{P}(X_n \in B_n)$ for all Borel sets $B_1, \dots, B_n \subseteq \mathbb{R}$,*

(iii) *the law of $X$ is a product of the laws of $X_1, \dots, X_n$, i.e., $\mathbb{P}_X = \mathbb{P}_{X_1} \times \cdots \times \mathbb{P}_{X_n}$,*

(iv) *the characteristic function of $X$ is the product of the characteristic functions of $X_1, \dots, X_n$, that is,*

$$\mathbb{E}[e^{iu \cdot X}] = \mathbb{E}[e^{iu_1 X_1}] \cdots \mathbb{E}[e^{iu_n X_n}]$$

*for $u = (u_1, \dots, u_n) \in \mathbb{R}^n$.*

*Idea of proof.* (i) $\iff$ (ii): By definition.

(iii) $\implies$ (ii): Clear.

(ii) $\implies$ (iii): Follows from fact: if two Borel probability measures $\nu, \mu$ on $\mathbb{R}^n$ agree on sets of form $B_1 \times \cdots \times B_n$, $B_i$ Borel, then $\nu = \mu$.

(iii) $\implies$ (iv): Clear.

(iv) $\implies$ (iii): Follows from fact that a measure is uniquely determined by its Fourier transform. □

**Corollary 8.9.** *If $X, Y$ are integrable and independent, then*

$$\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y].$$

*Proof.* Let $Z = (X, Y) : \Omega \to \mathbb{R}^2$.

$$\begin{aligned} \mathbb{E}[XY] &= \int_{\mathbb{R}^2} xy \, d\,\mathbb{P}_Z(x,y) \\ &= \int_{\mathbb{R}^2} xy \, d\,\mathbb{P}_X(x)\mathbb{P}_Y(y) \qquad \text{(Theorem 8.8)} \\ &= \Big(\int_{\mathbb{R}} x \, d\,\mathbb{P}_X(x)\Big)\Big(\int_{\mathbb{R}} y \, d\,\mathbb{P}_Y(y)\Big) = \mathbb{E}[X] \cdot \mathbb{E}[Y]. \end{aligned}$$

□

**Lemma 8.10. (Borel-Cantelli-II)** *Let $A_n$, $n \in \mathbb{N}$, be independent events. If $\sum_{n=1}^\infty \mathbb{P}(A_n) = \infty$, then*

$$\mathbb{P}(A_{n,\text{i.o.}}) = 1.$$

*Proof.* Note that $e^{-x} \geq 1 - x$ for $x \in [0,1]$. So

$$\begin{aligned} \mathbb{P}(\textstyle\bigcup_{n=k}^N A_n) &= 1 - \mathbb{P}(\textstyle\bigcap_{n=k}^N A_n^c) = 1 - \textstyle\prod_{n=k}^N (1 - \mathbb{P}(A_n)) \qquad \text{(independence)} \\ &\geq 1 - \prod_{n=k}^N e^{-\mathbb{P}(A_n)} = 1 - e^{-\sum_{n=k}^N \mathbb{P}(A_n)} \to 1 \qquad \text{as} \quad N \to \infty. \end{aligned}$$

So

$$\mathbb{P}(\textstyle\bigcup_{n=k}^\infty A_n) = 1,$$

and

$$\mathbb{P}(A_{n,\text{i.o.}}) = \mathbb{P}(\textstyle\bigcap_{k=1}^\infty \bigcup_{n=k}^\infty A_n) = \lim_{k\to\infty} \mathbb{P}(\textstyle\bigcup_{n=k}^\infty A_n) = 1.$$

□

**Lemma 8.11.** *Let $X, Y : \Omega \to \mathbb{R}^n$ be random variables, let $Z = X + Y$. Then*

$$\mathbb{P}_Z = \mathbb{P}_X * \mathbb{P}_Y \qquad \text{(convolution)}$$

*and*

$$\phi_Z(u) := \mathbb{E}[e^{iu\cdot Z}] = \phi_X(u) \cdot \phi_Y(u), \qquad \text{for} \quad u \in \mathbb{R}^n.$$

*Proof.* Let $\pi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\pi(x, y) = x+y$. Then $\mathbb{P}_Z = \pi_* \mathbb{P}_{(X,Y)}$. Since $X, Y$ are independent, $\mathbb{P}_{(X,Y)} = \mathbb{P}_X \times \mathbb{P}_Y$. So if $A \subseteq \mathbb{R}^n$ is a Borel set, then

$$\mathbb{P}_Z(A) = \pi_* \mathbb{P}_{(X,Y)}(A) = \int \chi_A * \pi \, d\,\mathbb{P}_{(X,Y)} = \int \chi_A(x+y) d\,\mathbb{P}_X(x) \mathbb{P}_Y(y) = \int \chi_A d\,\mathbb{P}_X * \mathbb{P}_Y.$$

Hence $\mathbb{P}_Z = \mathbb{P}_X * \mathbb{P}_Y$.

$$\phi_Z(u) = \mathbb{E}[e^{iu\cdot(X+Y)}] = \mathbb{E}[e^{iu\cdot X} e^{iu\cdot Y}] \overset{\text{ind.}}{=} \mathbb{E}[e^{iu\cdot X}] \cdot \mathbb{E}[e^{iu\cdot Y}] = \phi_X(u) \cdot \phi_Y(u). \qquad \square$$

### 8.12. Gaussian random variables

Let $X : \Omega \to \mathbb{R}$ be a real-valued random variable. Then $X$ is *Gaussian* with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$ if its distribution is given by

$$d\,\mathbb{P}_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(-\frac{(x-\mu)^2}{2\sigma^2}\Big) dx. \qquad \text{(Gaussian or normal distribution)}$$

We write $X \sim \mathcal{N}(\mu, \sigma^2)$.

$X$ is *standard* Gaussian or normal if $X \sim \mathcal{N}(0, 1)$, i.e.,

$$d\,\mathbb{P}_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

If $X \sim \mathcal{N}(0, 1)$, then $\mathbb{E}[X] = \mu$ and $\mathrm{Var}[x] = \sigma^2$, and $\sigma = \mathrm{Var}[x]^{1/2}$ the *standard deviation.*

Characteristic function: if $X \sim \mathcal{N}(\mu, \sigma)$, then

$$\phi_X(u) = \exp\Big(-\frac{1}{2}\sigma^2 u^2 + in\mu\Big).$$

If $X \sim \mathcal{N}(\mu_1, \sigma_1^2)$, $Y \sim \mathcal{N}(\mu_2, \sigma_2^2)$, and $X, Y$ are independent, then

$$Z = X + Y \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2).$$

*Proof.* $\phi_Z(u) = \phi_X(u) \cdot \phi_Y(u) = \exp\Big(-\frac{1}{2}(\sigma_1^2 + \sigma_2^2)u^2 + iu(\mu_1 + \mu_2)\Big)$. $\square$

It is convenient to consider a random variable $x$ such that $X = \mu$ a.s. as a "generalized" Gaussian, where $\sigma^2 = 0$. Namely,

$$\mathbb{P}_X = \delta_\mu, \qquad \phi_X(u) = \exp(-iu\mu) = \exp\Big(-\frac{1}{2} 0 u^2 + iu\mu\Big).$$

**Definition.** A random variable $X = (X_1, \dots, X_n) : \Omega \to \mathbb{R}^n$ is a (generalized, vector valued) Gaussian, if

$$\phi_X(u) = \mathbb{E}[e^{iu\cdot X}] = \exp\Big(-\frac{1}{2} u^t C u + iu \cdot \mu\Big) \qquad \text{for} \quad u \in \mathbb{R}^n,$$

where $\mu \in \mathbb{R}^n$ and $C$ is a positive semi-defined $n \times n$-matrix.

Let

$$\mathrm{Cov}(X,Y) := \mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])]$$

be the *covariance* of $X, Y$. Then $C$ is the covariance matrix of $X$, i.e., $C=(c_{ij})$, where

$$c_{ij} = \mathrm{Cov}(X_i, X_j).$$

$X$ is Gaussian iff $X = BY$, where $B$ is a $n\times n$-matrix and $Y=(Y_1,\dots,Y_n)$ such that $Y_1,\dots,Y_n$ are real-valued independent generalized Gaussians iff $X = DZ + a$, where $a\in\mathbb{R}^n$, $D$ is a $n\times k$-matrix, $Z=(Z_1,\dots,Z_k)$, $Z_1,\dots,Z_k$ is independent Gaussians.

Let $Y = AX$, where $A$ is a $n\times k$-matrix, $X:\Omega\to\mathbb{R}^n$, $Y:\Omega\to\mathbb{R}^k$. If $X$ is Gaussian, then $Y$ is Gaussian.

*Proof.*

$$\begin{aligned}\phi_Y(v) &= \mathbb{E}[e^{iv\cdot Y}] = \mathbb{E}[e^{iv\cdot AX}] = \mathbb{E}[e^{iA^tv\cdot X}]\\ &= \phi_X(A^tv) = \exp\Big(-\frac{1}{2}(A^tv)^tC(A^tv) + i(A^tv)\cdot\mu\Big)\\ &= \exp\Big(-\frac{1}{2}v^t(ACA^t)v + iv\cdot A\mu\Big).\end{aligned}$$

So $\mu' = A\mu$, $C' = ACA^t$. □

If $X:\Omega\to\mathbb{R}^n$ has a *multi-normal distribution* given by

$$d\,\mathbb{P}_X(x) = \frac{|A|^{1/2}}{(2\pi)^{n/2}}\exp\Big(-\frac{1}{2}(x-\mu)^tA(x-\mu)\Big),$$

where $\mu\in\mathbb{R}^n$, and $A$ is a positive defined $n\times n$-matrix, then $X$ is Gaussian and

$$\phi_X(u) = \exp\Big(-\frac{1}{2}u^tCu + i(u\cdot\mu)\Big),$$

where $C = A^{-1}$.

### 8.13. Modes of convergence of random variables

Let $X_n$, $n\in\mathbb{N}\cup\{\infty\}$, be real (or vector valued) random variables.

i) $X_n\to X_\infty$ a.s. (almost surely) iff

$$\mathbb{P}(X_n\to X_\infty) = \mathbb{P}(\{\omega\in\Omega : X_n(\omega)\to X_\infty(\omega)\}) = 1,$$

iff $X_n\to X_\infty$ for a.e. $\omega\in\Omega$.

ii) $X_n\to X_\infty$ *in probability* iff

$$\lim_{n\to\infty}\mathbb{P}(|X_n - X_\infty|\ge\varepsilon) = 0 \qquad \text{for all} \quad \varepsilon>0.$$

(equivalent to "convergence in measure".)

iii) $X_n\to X_\infty$ in $L^p$, $p\ge 1$, iff

$$\mathbb{E}[|X_n - X_\infty|^p]\to 0,$$

equivalently

$$\int_\Omega |X_n(\omega) - X_\infty(\omega)|^p d\,\mathbb{P}(\omega)\to 0.$$

$$\begin{array}{ccc} X_n \to X_\infty \text{ a.s.} & \overset{1}{\Longrightarrow} & \\ 3 \Uparrow \text{ subseq.} & & X_n \to X_\infty \text{ in probability} \\ X_n \to X_\infty \text{ in } L^p & \overset{2}{\Longrightarrow} & \end{array}$$

*Proof.* (easy) e.g. 1: Fix $\varepsilon > 0$, define

$$E_n = \{\omega \in \Omega : |X_n(\omega) - X_\infty(\omega)| \geq \varepsilon\}.$$

Then $X_n \to X_\infty$ a.s. implies

$$0 = \mathbb{P}(E_{n,\text{i.o.}}) = \mathbb{P}(\textstyle\bigcap_n \bigcup_{k\geq n} E_k) = \lim_{n\to\infty} \mathbb{P}(\bigcup_{k\geq n} E_k) \geq \limsup_{n\to\infty} \mathbb{P}(E_n).$$ □

**Lemma 8.14.** *Let $X_n$ be $\mathbb{R}^d$-valued Gaussian random variables, $n \in \mathbb{N}$, $X_n \to X_\infty$ in probability. Then $X_\infty$ is $\mathbb{R}^d$-valued Gaussian.*

*Proof.* (outline) 1. If $X_n \to X_\infty$ in probability, then

$$\phi_{X_n}(u) \to \phi_{X_\infty}(u) \qquad \text{locally uniformly on } \mathbb{R}^d. \tag{23}$$

In fact,

$$|e^{iu\cdot X_n} - e^{iu\cdot X_\infty}| \leq |u \cdot X_n - u \cdot X_\infty| \leq |u| \cdot |X_n - X_\infty|.$$

So

$$|\phi_{X_n}(u) - \phi_{X_\infty}(u)| \leq \mathbb{E}[|e^{iu\cdot X_n} - e^{iu\cdot X_\infty}|] \leq |u|\delta + 2\mathbb{P}(|X_n - X_\infty| \geq \delta) \leq \varepsilon$$

for $n$ large. So (23) follows.

2. $X_n$ Gaussian, so

$$\phi_{X_n}(u) = \exp\Big(-\frac{1}{2}u^t C_n u + iu \cdot \mu_n\Big),$$

where $C_n \geq 0$ and $\mu_n \in \mathbb{R}^d$. If

$$\phi_{X_n}(u) \to \phi_{X_\infty}(u) \qquad \text{locally uniformly,}$$

then $\phi_{X_\infty}$ has the same form, i.e.,

$$\phi_{X_\infty}(u) = \exp\Big(-\frac{1}{2}u^t C u + iu \cdot \mu\Big),$$

where $C \geq 0$ and $\mu \in \mathbb{R}^d$. □

**Lemma 8.15.** *Let $X_1, \ldots, X_n$ be real-valued random variables with joint Gaussian distribution (i.e., $X = (X_1, \ldots, X_n)$ is $\mathbb{R}^n$-valued Gaussian random variable). Then $X_1, \ldots, X_n$ are independent iff they are pairwise uncorrelated, i.e.,* $\text{Cov}(X_i, X_j) = 0$ *for $i, j = 1, \ldots, n$, $i \neq j$.*

*Proof.* "$\Longrightarrow$" Clear:

$$\text{Cov}(X_i, X_j) = \mathbb{E}[(X_i - \mathbb{E}[x_i])(X_j - \mathbb{E}[X - j])] \overset{\text{ind.}}{=} \mathbb{E}[X_i - E[X_i]] \cdot \mathbb{E}[X_j - \mathbb{E}[X_j]] = 0.$$

"$\Longleftarrow$" Since $X$ Gaussian,

$$\phi_X(u) = \exp\Big(-\frac{1}{2}u^t C u + iu \cdot \mu\Big), \qquad u \in \mathbb{R}^n,$$

where $C = (C_{ij})$ is the covariance matrix. So $c_{ij} = \mathrm{Cov}(X_i, X_j)$, $i, j = 1, \ldots, n$.

By assumption, $c_{ij} = 0$ for $i \neq j$, and so $C$ is a diagonal matrix. Hence,

$$\phi_X(u) = \phi_{X_1}(u_1) \cdots \phi_{X_n}(u_n)$$

for $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$. This shows that $X_1, \ldots, X_n$ are independent by Theorem 8.8. □

### 8.16. Stochastic processes

A *stochastic process* in $\mathbb{R}^n$ is a collection $\{X_t\}_{t \in T}$ of random variables defined on a probability space $(\Omega, \mathscr{A}, \mathbb{P})$, where $T \subseteq \mathbb{R}$ the parameter set of "times".

If $T = \mathbb{N}_0, \mathbb{N}$, it is a "discrete time stochastic process", which is a sequence of random variables: $X_1, X_2, \ldots, X_n, \ldots$.

If $T = [0, \infty), [a, b]$ etc., it is a "continuous time stochastic process".

If $t \in T$ fixed, $\omega \mapsto X_t(\omega)$ is a random variable on $\Omega$. If $\omega$ fixed, $t \in T \mapsto X_t(\omega)$ is a sample path of the stochastic process.

**Definition 8.17. (Brownian motion)** A real-valued stochastic process $\{B_t\}_{t \in [0,\infty)}$ is called a (version of) *Brownian motion* if the following conditions are true:

(i) the process is a Gaussian process, i.e., for all $n \in \mathbb{N}$, $0 \leq t_1 < \cdots < t_n$, the random variables $B_{t_1}, \ldots, B_{t_n}$ have a joint Gaussian distribution.

(ii) $B_t$ for $t \in [0, \infty)$ is *centered*, i.e., $\mathbb{E}[B_t] = 0$.

(iii) $\mathrm{Cov}(B_t, B_s) = \mathbb{E}[B_t B_s] = s \wedge t$, $s, t \in [0, \infty)$.

(iv) sample paths $t \mapsto B_t$ are continuous a.s., i.e., $t \mapsto B_t(\omega)$ is continuous for a.e. $\omega$.

**Remark 8.18.** Let $\{B_t\}_{t \in [0,\infty)}$ be a Brownian motion.

1) $\mathbb{E}[B_t] = 0$, $\mathrm{Var}(B_t) = \mathrm{Cov}(B_t, B_t) = t$ for $t \geq 0$. So $B_t \sim \mathcal{N}(0, t)$ for $t > 0$, $B_0 = 0$ a.s.. Brownian motion starts at 0 from time 0 a.s..

2) Brownian motion has "independent increments". If $t_1 < t_2 < \ldots < t_n$, then

$$X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}} \tag{24}$$

are independent Gaussian random variables.

$$X_{t_k} - X_{t_{k-1}} \sim \mathcal{N}(0, t_k - t_{k-1}).$$

Indeed, the random variables in (24) are joint Gaussian, centered, and for $k < l$, $t_{k-1} < t_k \leq t_{l-1} < t_l$,

$$\begin{aligned}
\mathrm{Cov}(X_{t_k} - X_{t_{k-1}}, X_{t_l} - X_{t_{l-1}}) &= \mathbb{E}[(X_{t_k} - X_{t_{k-1}})(X_{t_l} - X_{t_{l-1}})] \\
&= t_k \wedge t_l - t_{k-1} \wedge t_l - t_k \wedge t_{l-1} + t_{k-1} \wedge t_{l-1} \\
&= t_k - t_{k-1} - t_k + t_{k-1} = 0.
\end{aligned}$$

So by Lemma 8.15, the random variables in (24) are independent.

### 8.19. Hilbert space bases

Let $H$ be a separable real Hilbert space. A sequence $\{x_n\}_{n \in \mathbb{N}}$ is called a *complete orthonormal system* or a *Hilbert space basis* if

i) the vectors are orthonormal, i.e., $(x_i, x_j) = \delta_{ij}$, $i, j \in \mathbb{N}$,

ii) if $x \in H$ and $(x, x_n) = 0$ for all $n \in \mathbb{N}$, then $x = 0$.

In this case,

$$x = \sum_{n=1}^{\infty} (x, x_n) x_n,$$

$$\|x\|^2 = \sum_{n=1}^{\infty} |(x, x_n)|^2, \quad (x, y) = \sum_{n=1}^{\infty} (x, x_n)(y, x_n). \qquad \text{(Parseval's identities)}$$

Equivalent to ii) is
ii′) the set $S$ of all (finite) linear combinations of the vectors $x_1, x_2, \ldots, x_n, \ldots$ is dense in $H$.

**Example.** Let $H = L^2[0,1]$, with inner product $(f,g) = \int_0^1 f(x)g(x)dx$. The Hilbert space bases:

1. trigonometric functions basis

$$\frac{1}{\sqrt{2}} \cos(2\pi n x), \qquad \frac{1}{\sqrt{2}} \sin(2\pi n x), \qquad n \in \mathbb{N}.$$

2. Haar basis

$$\varphi_{n,k}(x) := \begin{cases} 1 & \\ -1 & \text{for} \quad x \in \\ 0 & \end{cases} \begin{matrix} [k/2^n, (k+1/2)/2^n), \\ [(k+1/2)/2^n, (k+1)/2^n), \\ \text{else,} \end{matrix}$$

where $n \in \mathbb{N}_0$, $k = 0, 1, \ldots, 2^n - 1$. $\varphi_{-1,0} \equiv 1$. Denote $I$ the set of indices.

Obviously, $\varphi_{n,k} \in L^2[0,1]$, pairwise orthogonal.

$$\|\varphi_{n,k}\|^2 = \int_0^1 \varphi_{n,k}(x)^2 dx = \frac{1}{2^n}, \qquad n \in \mathbb{N}_0.$$

Set

$$\psi_{n,k} = 2^n \varphi_{n,k}, \qquad \psi_{-1,0} \equiv 1.$$

Then $\{\psi_{n,k}\}_{(n,k)\in I}$ forms an orthonormal system. Its linear combinations are dense in $L^2[0,1]$ (because step functions on dyadic intervals are). So $\{\psi_{n,k}\}_{(n,k)\in I}$ is a Hilbert space basis of $L^2[0,1]$.

If $\{x_n\}$ is an orhtonormal system, then

$$\sum_{n=1}^{\infty} a_n x_n \text{ converges} \quad \text{iff} \quad \sum_{n=1}^{\infty} a_n^2 < \infty.$$

In fact, it follows from the Cauchy criterion since the partial sum $s_n = \sum_{k=1}^n a_k x_k$ satisfies

$$\|s_n - s_m\|^2 = \sum_{k=m+1}^{n} a_k^2, \qquad n \geq m.$$

**8.20. Construction of Brownian motion**

**1. Brownian motion on $T = [0,1]$.**

Let $Z_n$, $n \in \mathbb{N}$, be i.i.d. random variables, i.e., independent, identically distributed random variables on the same probability space $(\Omega, \mathscr{A}, \mathbb{P})$, and $Z_n \sim \mathcal{N}(0,1)$. For example, let $\tilde{\Omega} = (\mathbb{R}, \mathscr{B}, \mu)$, where $\mathscr{B}$ is the Borel $\sigma$-algebra, and

$$d\mu(x) = \frac{1}{\sqrt{2}} e^{-x^2/2} dx.$$

Set $\Omega = \tilde{\Omega}^{\mathbb{N}}$, and $Z_n =$ the projection onto the $n$-th coordinate.

$Z_n$, $n \in \mathbb{N}$, forms an orthonormal system in $L^2(\Omega)$. In fact,

$$\int_\Omega Z_n(\omega) Z_k(\omega) d\,\mathbb{P}(\omega) = \operatorname{Cov}(Z_n, Z_k) = \delta_{nk}, \qquad n, k \in \mathbb{N}.$$

Let $\psi_n$, $n \in \mathbb{N}$, be a Hilbert space basis of $L^2[0,1]$. Let

$$f_n(t) = \int_0^t \psi_n(u) du = (\psi_n, \chi_{[0,t]}), \qquad \text{inner product in} \quad L^2[0,1].$$

Define

$$B_t = \sum_{n=1}^{\infty} f_n(t) Z_n, \qquad \text{for} \quad t \in [0,1].$$

i) For each $t \in [0,1]$, the sum converges in $L^2[0,1]$, equivalently,

$$\sum_{n=1}^{\infty} f_n(t)^2 = \sum_{n=1}^{\infty} (\psi_n, \chi_{[0,t]})^2 \overset{*}{=} \|\chi_{[0,t]}\|^2 = t < \infty. \qquad (*\text{ Parseval})$$

ii) Each $B_t$ is a Gaussian; actually, for $t_1 < t_2 < \cdots < t_m$, $B_{t_1}, B_{t_2}, \dots, B_{t_m}$ have a joint Gaussian distribution.

In fact,

$$B_t^n := \sum_{k=1}^{n} f_k(t) Z_k$$

is Gaussian (linear combination of Gaussians), and $B_t^n \to B_t$ as $n \to \infty$ in $L^2(\Omega)$. So $B_t$ is Gaussian by Lemma 8.14.

Similarly, $(B_{t_1}^n, B_{t_2}^n, \dots, B_{t_m}^n)$ have a joint Gaussian distribution, and $(B_{t_1}^n, B_{t_2}^n, \dots, B_{t_m}^n) \to (B_{t_1}, B_{t_2}, \dots, B_{t_m})$ as $n \to \infty$ in $L^2(\Omega, \mathbb{R}^m)$. So $(B_{t_1}, B_{t_2}, \dots, B_{t_m})$ have a joint Gaussian distribution.

iii) $B_t$ is centered.

$$\mathbb{E}[B_t] = \int_\Omega B_t(\omega) d\,\mathbb{P}(\omega) = \lim_{n\to\infty} \int_\Omega B_t^n(\omega) d\,\mathbb{P}(\omega) = \lim_{n\to\infty} \sum_{k=1}^{n} f_k(t) \mathbb{E}[Z_k] = 0,$$

because $Z_n$, $n \in \mathbb{N}$, is centered.

iv)

$$\begin{aligned}
\operatorname{Cov}(B_s, B_t) &= \int_\Omega B_s(\omega) B_t(\omega) d\,\mathbb{P}(\omega) = \lim_{n\to\infty} \int_\Omega B_s^n(\omega) B_t^n(\omega) d\,\mathbb{P}(\omega) \\
&= \lim_{n\to\infty} \sum_{k,l=1}^{n} f_k(s) f_l(t) \operatorname{Cov}(Z_k, Z_l) \overset{*}{=} \sum_{k=1}^{\infty} f_k(s) f_k(t) \qquad (*\operatorname{Cov}(Z_k, Z_l) = \delta_{kl}) \\
&= \sum_{k=1}^{\infty} (\psi_k, \chi_{[0,s]})(\psi_k, \chi_{[0,t]}) \overset{**}{=} (\chi_{[0,s]}, \chi_{[0,t]}) = s \wedge t \qquad (** \text{ Parseval})
\end{aligned}$$

To check the continuity of $t \mapsto B_t(\omega)$ for a.e. $\omega \in \Omega$, we choose the Haar basis for the Hilbert space basis of $L^2[0,1]$. Let $\{\psi_{n,k}\}_{(n,k)\in I}$ be the Haar basis of $L^2[0,1]$, let

$$f_{n,k}(t) = \int_0^t \psi_{n,k}(s) ds.$$

Then $f_{n,k}$ is Lipschitz with Lipschitz constant $\operatorname{Lip}(f_{n,k}) = 2^{n/2}$.

$$\|f_{n,k}\|_\infty \le \frac{1}{2}\frac{1}{2^n} 2^{n/2} = \frac{1}{2^{n/2+1}} \lesssim \frac{1}{2^{n/2}}.$$

**Claim.** Let $\{Z_{n,k}\}_{(n,k)\in I}$ be i.i.d. standard normal random variables. Then for a.e. $\omega \in \Omega$, the series

$$B_t(\omega) = Z_{-1,0} f_{-1,0}(t) + \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} Z_{n,k}(\omega) f_{n,k}(t) \tag{25}$$

converges uniformly in $t$ (and hence represents a continuous function in $t$).

*Proof.* Note that

$$\frac{2}{\sqrt{2\pi}} \int_a^\infty e^{-x^2/2} dx \le e^{-a^2/2} \qquad \text{for} \quad a > 0.$$

So

$$\mathbb{P}(|Z| > a) \le e^{-a^2/2} \qquad \text{for} \quad a \ge 0$$

if $Z \sim \mathcal{N}(0,1)$. Denote

$$A_{n,k} = \left\{ |Z_{n,k}| > 2\sqrt{\log(2^{n/2} n)} \right\}.$$

Then

$$\mathbb{P}(A_{n,k}) \le e^{-2\log(2^{n/2} n)} = \frac{1}{2^n n^2}.$$

So

$$\sum_{n=1}^{\infty} \sum_{k=0}^{2^n-1} \mathbb{P}(A_{n,k}) \le \sum_{n=1}^{\infty} \frac{2^n}{2^n n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

By Borel-Cantelli-I, we have $\mathbb{P}(A_{n,k,\text{i.o.}}) = 0$, i.e., for a.e. $\omega \in \Omega$, we have

$$|Z_{n,k}(\omega)| \le 2\sqrt{\log(2^{n/2} n)} \lesssim \sqrt{n} \tag{26}$$

for all sufficiently large $n$ (depending on $\omega$), say for $n \ge N(\omega)$.

For such $\omega$,

$$\sum_{n=N(\omega)}^{\infty}\left|\sum_{k=0}^{2^n-1} Z_{n,k}(\omega)f_{n,k}(t)\right| \lesssim \sum_{n=N(\omega)}^{\infty}\frac{\sqrt{n}}{2^{n/2}} < \infty.$$

So series (25) represents a continuous function in $t$ by the Weierstrass M-test.

Actually, for such $\omega$,

$$g_n(t) = \sum_{k=0}^{2^n-1} Z_{n,k}(\omega)f_{n,k}(t) \qquad (g_{-1}(t) = Z_{-1,0}(\omega)f_{-1,0}(t))$$

is $L_n$-Lipschitz with $L_n \lesssim \sqrt{n}2^{n/2}$ for all $n \geq N(\omega)$; so by adjusting constants wlog for all $n \geq 1$. Moreover, $\|g_n\|_\infty \lesssim \sqrt{n}2^{n/2}$ for all $n \geq N(\omega)$, wlog for all $n \geq 1$.

Suppose $\omega$ is "good" so that it satisfies (26). Let $s, t \in [0,1]$. Pick suitable $N = N(s,t) \in \mathbb{N}$. Then

$$\begin{aligned}
|B_s(\omega) - B_t(\omega)| &\leq \sum_{n=-1}^{\infty} |g_n(s) - g_n(t)| \\
&\leq \sum_{n=-1}^{N} L_n|s-t| + \sum_{n=N+1}^{\infty} 2\|g_n\|_\infty \\
&\overset{\omega}{\lesssim} \Big(1 + \sum_{n=1}^{N}\sqrt{n}2^{n/2}\Big)|s-t| + \sum_{n=N+1}^{\infty}\sqrt{n}2^{-n/2} \\
&\overset{\omega}{\lesssim} \sqrt{N}2^{N/2}|s-t| + \sqrt{N}2^{-N/2}.
\end{aligned}$$

Pick $N = N(s,t)$ such that $2^{N/2}|s-t| = 2^{-N/2}$, equivalently $|s-t| \sim 2^{-N}$, equivalently

$$N = \log_2\frac{1}{|s-t|} \sim \log\frac{1}{|s-t|}.$$

Then

$$|B_s(\omega) - B_t(\omega)| \lesssim |s-t|^{1/2}\sqrt{\log\frac{1}{|s-t|}}. \qquad \square$$

**Conclusion.** For a.e. $\omega$, there exists $M(\omega) \geq 0$, such that

$$|B_s(\omega) - B_t(\omega)| \leq M(\omega)|s-t|^{1/2}\sqrt{\log\frac{1}{|s-t|}}.$$

Almost surely, the sample path $t \mapsto B_t(\omega)$ has modulus of continuity

$$\omega(\delta) = C\delta^{1/2}\sqrt{\log(1/\delta)}.$$

So for every $\varepsilon > 0$, $t \mapsto B_t(\omega)$ is $(1/2-\varepsilon)$-Hölder almost surely.

**2. Brownian motion on** $[0,\infty)$**.**

**Idea.** Let a Brownian motion run until time 1, start a "new" Brownian motion at endpoint, let it run until time 2, etc.

Let $B_t^n$, $n \in \mathbb{N}_0$, be independent copies of Brownian motion on $[0,1]$. Define

$$B_t(\omega) = \sum_{k=0}^{\lfloor t \rfloor - 1} B_1^k(\omega) + B_{t-\lfloor t \rfloor}^{\lfloor t \rfloor}(\omega).$$

$$(\text{ e.g.} \quad B_{1.5}(\omega) = B_1^0(\omega) + B_{0.5}^1(\omega).)$$

Then $\{B_t\}_{t\in[0,\infty)}$ is a Gaussian process, $B_t$ centered, and for $s \le t$,

$$\begin{aligned}\operatorname{Cov}(B_s, B_t) &= \mathbb{E}\Big[\Big(\sum_{k=0}^{\lfloor s \rfloor - 1} B_1^k + B_{s-\lfloor s \rfloor}^{\lfloor s \rfloor}\Big)\Big(\sum_{k=0}^{\lfloor t \rfloor - 1} B_1^k + B_{t-\lfloor t \rfloor}^{\lfloor t \rfloor}\Big)\Big] \\ &= \sum_{k=0}^{\lfloor s \rfloor - 1} 1 + (s - \lfloor s \rfloor) = s = s \wedge t.\end{aligned}$$

For each $n \in \mathbb{N}_0$, $t \mapsto B_t^n(\omega)$ on $[0,1]$ is continuous a.s., so for a.e. $\omega$, $t \mapsto B_t^n(\omega)$ are continuous for all $n \in \mathbb{N}_0$. Hence $t \mapsto B_t(\omega)$ is continuous a.s..

### 8.21. $\pi$-systems

Let $X$ be a set, $\mathscr{S}$ be a family of subsets of $X$. $\mathscr{S}$ is called a *$\pi$-system* if $A \cap B \in \mathscr{S}$ whenever $A, B \in \mathscr{S}$. (i.e., a $\pi$-system is "stable" under the finite intersection.)

**Facts.** 1) Let $\mathscr{S}$ be a $\pi$-system, let $\mathscr{A} = \sigma(\mathscr{S})$ be a $\sigma$-algebra generated by $\mathscr{S}$, and let $\mu, \nu$ be probability measures on $\mathscr{A}$. If $\mu(A) = \nu(A)$ for all $A \in \mathscr{S}$, then $\mu = \nu$. (i.e., $\mu(A) = \nu(A)$ for all $A \in \mathscr{A}$.) (Exercise!)

2) Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a probability space. Let $\mathscr{S}, \mathscr{T}$ be two $\pi$-systems, and let $\mathscr{B} = \sigma(\mathscr{S}), \mathscr{C} = \sigma(\mathscr{T}) \subseteq \mathscr{A}$. If $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ whenever $A \in \mathscr{S}$, $B \in \mathscr{T}$, then $\mathscr{B}$ and $\mathscr{C}$ are independent. (i.e., $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ for all $A \in \mathscr{B}$, $B \in \mathscr{C}$.) (Exercise!)

### 8.22. The space $X = C([0,\infty))$

Let

$$X := C([0,\infty)) = \{f : [0,\infty) \to \mathbb{R} \text{ continuous}\}$$

equipped with "topology of locally uniform convergence": $f_n \to f$ iff $f_n \to f$ locally uniformly on $\mathbb{R}$.

This is a metrizable topology: Let

$$d_n(f,g) = \sup_{x\in[0,n]\cap\mathbb{Q}} |f(x) - g(x)|, \qquad d(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(f,g)}{1 + d_n(f,g)}.$$

Then $d$ is a metric on $X$. $d(f_n, f) \to 0$ iff $f_n \to f$ locally uniformly on $[0,\infty)$. $(X,d)$ forms a separable space.

Let $\mathscr{B} = \mathscr{B}_X$, the Borel $\sigma$-algebra on $X$ (i.e., the smallest $\sigma$-algebra containing all open sets in $X$). We want to find $\pi$-system $\mathscr{S}$ such that $\mathscr{B} = \sigma(\mathscr{S})$.

For $t \in [0,\infty)$, let

$$\pi_t : X \to \mathbb{R}, f \mapsto f(t)$$

be the evaluation of time $t$. Let

$$\mathscr{S} = \{\pi_{t_1}^{-1}(B_1) \cap \cdots \cap \pi_{t_k}^{-1}(B_k) : k \in \mathbb{N}, t_1 < \cdots < t_k \text{ in } [0,\infty), B_1, \ldots, B_k \in \mathscr{B}_{\mathbb{R}}\}.$$

Obviously, $\mathscr{S}$ is a $\pi$-system!

**Claim.** $\sigma(\mathscr{S}) = \mathscr{B}$.

*Proof.* (Outline) 1. For $t \in [0,\infty)$, $\pi_t : X \to \mathbb{R}$ is continuous. So $\pi_t^{-1}(B) \in \mathscr{B}_X$ for each $B \in \mathscr{B}_{\mathbb{R}}$, and $\mathscr{S} \subseteq \mathscr{B}_{\mathbb{R}}$. Hence $\sigma(\mathscr{S}) \subseteq \mathscr{B}_X$.

2. $\mathscr{B}_X \subseteq \sigma(\mathscr{S})$.

Let $f_0 \in X$ be arbitrary. Then $f \mapsto |f(t) - f_0(t)|$ is $\sigma(\mathscr{S})$-measurable. So $f \mapsto d_n(f, f_0) = \sup_{t\in[0,n]} |f(t) - f_0(t)|$ is $\sigma(\mathscr{S})$-measurable, and $f \mapsto d(f, f_0) = \sum_{n=1}^{\infty} \frac{d_n(f,f_0)}{1+d_n(f,f_0)}$ is $\sigma(\mathscr{S})$-measurable. Thus, open balls $B_d(f_0, \varepsilon) = \{f : d(f, f_0) < \varepsilon\}$ are $\sigma(\mathscr{S})$-measurable. Since every open set in $X$ is a countable union of open balls, every open set is in $\sigma(\mathscr{S})$. Hence, $\mathscr{B}_X \subseteq \sigma(\mathscr{S})$. $\square$

Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a probability space.

**Claim.** $Z : \Omega \to X$ is measurable (w.r.t. $\mathscr{A}$ and $\mathscr{B}_X$) iff $Z_t := \pi_t \circ Z$ is measurable for each $t \in [0,\infty)$

$$\begin{array}{ccc} \Omega & \xrightarrow{Z} & X \\ & {\scriptstyle Z_t}\searrow \quad \swarrow{\scriptstyle \pi_t} & \\ & \mathbb{R} & \end{array}$$

*Proof.* "$\Longrightarrow$" If $Z$ is measurable, then $Z_t = \pi_t \circ Z$ is measurable, because $\pi_t$ is continuous.

"$\Longleftarrow$" Let $\mathscr{C} = \{A \in X : Z^{-1}(A) \in \mathscr{A}\}$. Then $\mathscr{C}$ is a $\sigma$-algebra. Let $B \subseteq \mathbb{R}$ be a Borel set, $t \in [0,\infty)$. Then

$$Z^{-1}(\pi_t(B)) = (\pi_t \circ Z)^{-1}(B) = Z_t^{-1}(B) \in \mathscr{A}$$

since $Z_t$ is measurable. So $\pi_t^{-1}(B) \in \mathscr{C}$. Hence, $\mathscr{S} \subseteq \mathscr{C}$ and $\sigma(\mathscr{S}) = \mathscr{B}_X \subseteq \mathscr{C}$. $\square$

**Theorem 8.23. (Canonical Brownian motion)** *Let $X = C([0,\infty))$, and $\mathscr{B} = \mathscr{B}_X$ the Borel $\sigma$-algebra on $X$. There exists a unique probability measure $W$ on $(X, \mathscr{B})$, called Wiener measure, with the following properties: if we define $B_t = \pi_t$, then $\{B_t\}_{t\in[0,\infty)}$ is a Brownian motion (on $\mathbb{R}$). More explicitly,*

i) *for $t_1 < \cdots < t_k$, the random variables $B_{t_1}, \ldots B_{t_k}$ have a joint Gaussian distribution. Equivalently, let $F \subseteq [0,\infty)$ be a finite set,*

$$\pi_F : X \to \mathbb{R}^F := \{\varphi : F \to \mathbb{R}\} \cong \mathbb{R}^{|F|}, \quad f \mapsto f|_F.$$

*Then*

$$\mu_F := (\pi_F)_*(W)$$

*is a "Gaussian measure" on $\mathbb{R}^F$.*

*Set $\mu_t := (\pi_t)_*(W)$.*

ii) *$B_t$ is centered, equivalent to*

$$\int_{\mathbb{R}} x d\mu_t(x) = 0, \qquad \textit{for each} \quad t \in [0,\infty).$$

iii) $\mathrm{Cov}(B_s, B_t) = s \wedge t$, *equivalent to*

$$\int_{\mathbb{R}^2} xy d\mu_{\{s,t\}}(x, y) = s \wedge t.$$

*Proof.* 1. *Uniqueness.* Suppose $W, \tilde{W}$ are two measures with the properties i)–iii). Then

$$(\pi_F)_*(W) = \mu_F = \tilde{\mu}_F = (\pi_F)_*(\tilde{W})$$

for each finite set $F \subseteq [0, \infty)$, because the Fourier transforms of $\mu_F, \tilde{\mu}_F$, and hence $\mu_F, \tilde{\mu}_F$ themselves are uniquely determined by i)–iii). This implies that for $t_1 < \ldots < t_k$, $F = \{t_1, \ldots, t_k\}$, and $B_1, \ldots, B_k \in \mathscr{B}_{\mathbb{R}}$, we have

$$\begin{aligned} W(\pi_{t_1}^{-1}(B_1) \cap \cdots \cap \pi_{t_k}^{-1}(B_k)) &= W(\pi_F^{-1}(B_1 \times \cdots \times B_k) \\ &= \mu_F(B_1 \times \cdots \times B_k) = \tilde{\mu}_F(B_1 \times \cdots \times B_k) \\ &= \tilde{W}(\pi_{t_1}^{-1}(B_1) \cap \cdots \cap \pi_{t_k}^{-1}(B_k)), \end{aligned}$$

i.e., $W(S) = \tilde{W}(S)$ for all $S \in \mathscr{S}$. Since $\sigma(\mathscr{S}) = \mathscr{B}_X$, we have $W = \tilde{W}$.

2. *Existence.* There exists Brownian motion $\{B_t\}_{t\in[0,\infty)}$ on some probability space $(\Omega, \mathscr{A}, \mathbb{P})$. By disregarding a set of measure 0, we may assume that $t \mapsto B_t(\omega)$ is continuous for *every* $\omega \in \Omega$. Define

$$B : \Omega \to X = C([0, \infty)), \quad \omega \mapsto (t \in [0, \infty) \mapsto B_t(\omega)).$$

Then for each $t \in [0, \infty)$, we have the commutation diagram

$$\begin{array}{ccc} \Omega & \xrightarrow{B} & X \\ & {\scriptstyle B_t} \searrow \quad \swarrow {\scriptstyle \pi_t} & \\ & \mathbb{R} & \end{array}$$

Since $B_t$ is measurable for each $t \in [0, \infty)$, the map $B$ is measurable (see two Claims in 8.22). Hence, $W := B_*(\mathbb{P})$ is a Borel probability measure on $X$, and if $F \subseteq [0, \infty)$ is finite, then

$$\mu_F = (\pi_F)_*(W) = (\pi_F)_*(B_*(\mathbb{P})) = (\pi_F \circ B)_*(\mathbb{P}) = (B_F)_*(\mathbb{P}),$$

where $B_F(\omega) := (B_{t_1}(\omega), \ldots, B_{t_k}(\omega))$. Hence $\{\pi_t\}_{t\in[0,\infty)}$ is a Brownian motion defined on $X$. □

### 8.24. Brownian motion on $\mathbb{R}^n$

A $\mathbb{R}^n$-valued stochastic process $\{B_t\}_{t\in[0,\infty)}$ is called a (version of) *Brownian motion on* $\mathbb{R}^n$ if the following conditions are true:

(i) the process is an $\mathbb{R}^n$-valued Gaussian process, i.e., for all $k \in \mathbb{N}$, $t_1 < \ldots < t_k$, the $\mathbb{R}^{nk}$-valued random variable $(B_{t_1}, \ldots, B_{t_k})$ has a Gaussian distribution.

Let $B_t = (B_t^1, \ldots, B_t^n)$, where $B_t^i$ is real-valued.

(ii) $B_t^i$ is centered for $i \in \{1, \ldots, n\}$, i.e., $\mathbb{E}[B_t^i] = 0$, $t \in [0, \infty)$.

(iii) $\mathrm{Cov}(B_s^i, B_t^j) = \delta_{ij} s \wedge t$, $i \in \{1, \ldots, n\}$, $s, t \in [0, \infty)$.

(iv) sample paths $t \mapsto B_t(\omega)$ are continuous a.s..

**Remark 8.25.** (i) If $B_t = (B_t^1, \dots, B_t^n)$ is a Brownian motion on $\mathbb{R}^n$, then $B_t^1, \dots, B_t^n$ are independent Brownian motions on $\mathbb{R}$. Conversely, if $B_t^1, \dots, B_t^n$ are independent Brownian motions on $\mathbb{R}$, then $B_t = (B_t^1, \dots, B_t^n)$ is a Brownian motion on $\mathbb{R}^n$. (This proves existence!)

(ii) Uniqueness. One can show (as in Theorem 8.23) that there exists a unique Wiener measure $W$ on $X = C([0,\infty), \mathbb{R}^n) = \{f : [0,\infty) \to \mathbb{R}^n \text{ continuous}\}$ such that $\{\pi_t\}_{t\in[0,\infty)}$ is a Brownian motion, where $\pi_t : X \to \mathbb{R}^n$, $f \mapsto f(t)$. Described by "marginal" on $\mathbb{R}^{|F|\times n}$, where $F \subseteq [0,\infty)$ finite, $\mu_F := (\pi_F)_*(W)$, $\pi_F : X \to (\mathbb{R}^n)^F$, $f \mapsto f|_F$.

(iii) $B_t = (B_t^1, \dots, B_t^n)$ is an $\mathbb{R}^n$-valued Brownian motion iff $W_t := \lambda_1 B_t^1 + \cdots \lambda_n B_t^n$ is a 1-dimensional Brownian motion for each unit vector $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$.

"$\Longrightarrow$" Clear:

$$\operatorname{Cov}(W_s, W_t) = \lambda_1^2 s \wedge t + \cdots + \lambda_n^2 s \wedge t = s \wedge t.$$

"$\Longleftarrow$" Need fact: "Let $Z_1, \dots, Z_n$ be $\mathbb{R}^k$-valued random variables. Then they have a joint Gaussian distribution iff $\lambda_1 Z_1 + \cdots + \lambda_n Z_n$ is Gaussian for all $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. Details left as exercise!

**8.26. Basic properties of Brownian motion**

Let $\{B_t\}_{t\in[0,\infty)}$ be a Brownian motion on $\mathbb{R}^n$. Then the following processes are also Brownian motions.

(i) $W_t = B_{t+s} - B_s$ for fixed $s \in [0,\infty)$ (Markov property). That is, Brownian motion is memoryless!

(ii) $W_t = AB_t$, if $A$ is an orthogonal transformation.

(iii) $W_t = (1/a)B_{a^2t}$, $a > 0$ fixed (Brownian scaling).

(iv) $W_t = \begin{cases} B_0, & t = 0, \\ tB_{1/t}, & t > 0 \end{cases}$ (time inversion).

*Proof.* All processes $W_t$ in (i)–(iv) are Gaussian, and $W_t$ is centered. One checks covariance: for example in (iii) and (iv).

$$\operatorname{Cov}(W_s^i, W_t^j) = \operatorname{Cov}\Big(\frac{1}{a}B^i_{a^2s}, \frac{1}{a}B^j_{a^2t}\Big) = \frac{1}{a^2}\delta_{ij}(a^2s) \wedge (a^2t) = \delta_{ij} s \wedge t.$$
$$\operatorname{Cov}(W_s^i, W_t^j) = st \operatorname{Cov}(B^i_{1/s}, B^j_{1/t}) = st\, \delta_{ij} \frac{1}{s} \wedge \frac{1}{t} = \delta_{ij} t \wedge s, \qquad s,t > 0.$$

Almost sure continuity of sample paths are clear for (i)–(iii), and on $(0,\infty)$ for (iv) (up to measure 0). Continuity of $W_t$ at 0 is the following event:

$$A = \bigcap_{\substack{\varepsilon>0 \\ \varepsilon\in\mathbb{Q}}} \bigcup_{\substack{\delta>0 \\ \delta\in\mathbb{Q}}} \bigcap_{\substack{0<t<\delta \\ t\in\mathbb{Q}}} \{\omega : |W_t(\omega) - W_0(\omega)| < \varepsilon\}.$$

If we replace $W_t$ by $B_t$, then this is an almost sure event. Since $W_t$ and $B_t$ have the same marginals, it follows that $A$ is almost sure. (Note: this shows that $\lim_{s\to\infty} |B_s|/s = 0$ a.s..) $\square$

**8.27. The stochastic Loewner equation (SLE)**

Chordal Loewner equation: Let $\{f_t\}_{t\in[0,\infty)}$ be a normalized chordal Loewner chain.

$$f_t(z) = z - \frac{2t}{z} + \cdots \qquad \text{near} \quad \infty.$$

The Loewner-Kufarev equation gives

$$\frac{\partial f}{\partial t}(z,t) = V(z,t)\frac{\partial f}{\partial z},$$

where

$$V(z,t) = 2\int_{\mathbb{R}} \frac{d\nu_t(u)}{u-z},$$

$\nu_t$ is a probability measure with $\operatorname{supp}(\nu_t) \Subset \mathbb{R}$.

One obtains $\mathrm{SLE}_\kappa$, $\kappa \geq 0$, if one take a probabilistic driving term here

$$\nu_t = \delta_{\sqrt{\kappa}B_t(\omega)},$$

where $B_t$ is the 1-dimensional Brownian motion. Then

$$V(z,t) = \frac{2}{\sqrt{\kappa}B_t(\omega) - z}.$$

Depending on $\omega$, one gets a "random" Loewner chain and corresponding random hulls $A_t(\omega)$.

One is interested in these hulls, because they can be used to study many conformally invariant processes in the plane.

**Problems.**

1) What are the characterizing properties of SLE?

(i.i.d. increments, Markov (= memoryless) property, conformal invariance, etc.)

2) What are the techniques to study SLE?

(Martingales method, etc.)

# 9 Survey of martingale theory

### 9.1. Conditional expectation

**Example.** Random expectation in two stages: Assume roll two dices with outcomes $X_1, X_2 \in \{1,2,3,4,5,6\}$. Let $Z = X_1 + X_2$, $\Omega = \{1,\ldots,6\} \times \{1,\ldots,6\}$. Then $\mathbb{E}[Z] = 7$.

Suppose the outcome of $X_1$ is known (partial information). Then we have to adjust $\mathbb{E}[Z]$ depending on $X_1$:

$$\mathbb{E}[Z|X_1 = x] = x + 3.5 = X_1(\omega) + 3.5.$$

We get a new random variable $\mathbb{E}[Z|X_1]$.

**Theorem and Definition 9.2. (Conditional expectation)** *Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a probability space, let $X$ be a random variable with $\mathbb{E}[|X|] < \infty$, let $\mathscr{B} \subseteq \mathscr{A}$ be a $\sigma$-algebra. Then there exists a random variable $Y$ on $\Omega$ such that*

i) *$Y$ is $\mathscr{B}$-measurable.*

ii) *$\mathbb{E}[|Y|] < \infty$.*

iii) *for every $B \in \mathscr{B}$, we have*

$$\mathbb{E}[Y;B] = \int_B Y(\omega)d\,\mathbb{P}(\omega) = \int_B X(\omega)d\,\mathbb{P}(\omega) = \mathbb{E}[X;B]$$

*$Y$ is essentially unique determined: if $\tilde{Y}$ is another random variable with properties* i)–iii)*, then $\tilde{Y} = Y$ a.s..*

The random variable $Y$ is called a (version of) conditional expectation of $X$ for given $\mathscr{B}$, denoted by $\mathbb{E}[X|\mathscr{B}]$.

*Idea of proof.* Wlog $X \geq 0$. Define

$$\mu(B) := \int_B X(\omega) d\,\mathbb{P}(\omega), \qquad \text{for} \quad B \in \mathscr{B}.$$

Then $\mu \ll \mathbb{P}|\mathscr{B}$. So $\mu$ has a Radon-Nikidyn derivative $Y$ w.r.t. $\mathbb{P}|\mathscr{B}$. Then i)–iii) are evident.

Uniqueness is also clear:

$$\mathbb{E}[X|Z_1, \ldots, Z_m] = \mathbb{E}[X|\sigma(Z_1, \ldots, Z_m)]. \qquad \square$$

### 9.3. Properties of conditional expectation

Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a probability space, all random variables $X$ satisfies $\mathbb{E}[|X|] < \infty$. Let $\mathscr{B} \subseteq \mathscr{A}$ be a $\sigma$-algebra.

(i) If $Y = \mathbb{E}[X|\mathscr{B}]$, then $\mathbb{E}[Y] = \mathbb{E}[X]$.

(ii) If $X$ is $\mathscr{B}$ measurable, then $\mathbb{E}[X|\mathscr{B}] = X$ a.s..

(iii) Linearity.

(iv) If $X \geq 0$, then $\mathbb{E}[X|\mathscr{B}] \geq 0$.

(v) (Monotone a.s. convergence) If $X_n \geq 0$, $X_n \nearrow X$, then

$$\mathbb{E}[X_n|\mathscr{B}] \nearrow \mathbb{E}[X|\mathscr{B}] \quad \text{a.s..}$$

(vi) (Dominated convergence) If $|X_n(\omega)| \leq V(\omega)$, $\mathbb{E}[V] < \infty$, $X_n \to X$ a.s., then

$$\mathbb{E}[X_n|\mathscr{B}] \to \mathbb{E}[X|\mathscr{B}] \quad \text{a.s..}$$

(vii) (Jensen) If $\varphi : \mathbb{R} \to \mathbb{R}$ convex, $\mathbb{E}[|\varphi(X)|] < \infty$, then

$$\varphi(\mathbb{E}[X|\mathscr{B}]) \leq \mathbb{E}[\varphi(X)|\mathscr{B}] \quad \text{a.s..}$$

In particular, $\big|\mathbb{E}[X|\mathscr{B}]\big| \leq \mathbb{E}[|X|\,|\mathscr{B}]$ and $\big\|\mathbb{E}[X|\mathscr{B}]\big\|_p \leq \|X\|_p$, $p \geq 1$.

(viii) (Tower property) If $\mathscr{B}, \mathscr{C}$ are two $\sigma$-algebras satisfying $\mathscr{C} \subseteq \mathscr{B} \subseteq \mathscr{A}$, then

$$\mathbb{E}\big[\mathbb{E}[X|\mathscr{B}]|\mathscr{C}\big] = \mathbb{E}[X|\mathscr{C}].$$

(ix) If $Z$ is $\mathscr{B}$-measurable, then

$$\mathbb{E}[ZX|\mathscr{B}] = Z\mathbb{E}[X|\mathscr{B}].$$

(x) If $X$ and $\mathscr{B}$ are independent, then

$$\mathbb{E}[X|\mathscr{B}] = \mathbb{E}[X] \quad \text{a.s.} \qquad \text{(constant function)}$$

*Proof.* Mostly straight forward from the definitions:

(vii) Jensen: $\varphi(x) = \sup_{L \leq \varphi \text{ affine}} L(x)$, $L(X) = aX + b \leq \varphi(X)$. So

$$\begin{aligned} L(\mathbb{E}[X|\mathscr{B}]) &\leq \mathbb{E}[L(X)|\mathscr{B}] && \text{linearity} \\ &\leq E[\varphi(X)|\mathscr{B}] && \text{monotonicity} \end{aligned}$$

Taking sup over all $L$ gives

$$\varphi(\mathbb{E}[X|\mathscr{B}]) \leq \mathbb{E}[\varphi(X)|\mathscr{B}].$$

Incorrect proof! Because we take sup over an uncountable family. Can be corrected if we write $\varphi = \sup_{L_n \le \varphi} L_n$ for a countable collection $L_n$, $n \in \mathbb{N}$.

(vi) for dominated convergence, we need Fatou's lemma:

If $X_n \ge 0$, then

$$\mathbb{E}\big[\liminf_{n\to\infty} X_n | \mathscr{B}\big] \le \liminf_{n\to\infty} \mathbb{E}[X_n|\mathscr{B}]. \qquad \square$$

**Example 9.4.** Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a probability space, let $\{A_n\}_{n\in\mathbb{N}}$ be a countable partition of $\Omega$ with $A_n \in \mathscr{A}$, $\mathbb{P}(A_n) > 0$. Define

$$\mathscr{B} = \sigma(\{A_n\}_{n\in\mathbb{N}}) = \Big\{ \bigcup_{n\in S} A_n : S \subseteq \mathbb{N} \text{ countable} \Big\}.$$

Then

$$\mathbb{E}[X|\mathscr{B}] = \sum_{n\in\mathbb{N}} \frac{1}{\mathbb{P}(A_n)} \int_{A_n} X(\omega) d\,\mathbb{P}(\omega) \cdot \chi_{A_n}(\omega) = \sum_{n\in\mathbb{N}} \mathbb{E}[X; A_n]\chi_{A_n}.$$

Check definition!

**Example 9.5.** (Fair games and martingales) Two players I ($P_1$) and II ($P_2$) roll dice. Consider a zero-sum game: at each step, player I wins or losses 1 unit. Let $X_n$ be winnings of $P_1$ after $n$ rolls (corr. $-X_n$ be winnings of $P_2$ after $n$ rolls).

Game 1. $P_1$ wins if roll $\in \{1, 2\}$ (so losses if $\in \{3, 4, 5, \}$. A not fair game!

Game 2. $P_1$ wins if roll even. A fair game!

Game 3. $P_1$ wins if roll even, if one of players has won $\ge 100$ units, then game biased against player as in Game 1. Game 3 is a fair game ($\mathbb{E}[X_n] = 0$ for all $n$), but not fair at all times (or all situations).

How to modal a game that is "fair at all times": $\mathbb{E}[X_{n+1} - X_n] = 0$ (true if $\mathbb{E}[X_{n+1}] = \mathbb{E}[X_n] = 0$). The better is $\mathbb{E}[X_{n+1} - X_n | X_n = x] = 0$, whatever $x$.

Let $\mathscr{F}_n$ be a $\sigma$-algebra of events that will be known at time $n$ ($\mathbb{E}[X_n|\mathscr{F}_n] = X_n$). Then

$$\mathbb{E}[X_{n+1} - X_n|\mathscr{F}_n] = 0 \quad \text{equivalent to} \quad \mathbb{E}[X_{n+1}|\mathscr{F}_n] = X_n.$$

**Definition 9.6. (Martingales; discrete-time case)** Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a probability space with *filtration* given by $\sigma$-algebras $\mathscr{F}_n \subseteq \mathscr{A}$ for $n \in \mathbb{N}_0$, i.e., $\mathscr{F}_n \subseteq \mathscr{F}_{n+1}$ for $n \in \mathbb{N}_0$. Let $X = \{X_n\}_{n\in\mathbb{N}_0}$ be a sequence of random variables on $\Omega$. Then $X$ is called a *martingale* if

(i) $X_n$ is $\mathscr{F}_n$-measurable for $n \in \mathbb{N}_0$, and $X_n \in L^1$, i.e., $\mathbb{E}[|X_n|] < \infty$,

(ii) $\mathbb{E}[X_{n+1}|\mathscr{F}_n] = X_n$ (a.s.) for $n \in \mathbb{N}_0$.

If in (ii), we have $\le$ or $\ge$, then $X$ is called a supermartingale or submartingale, respectively. (Submartingale: tendency to increase, supermartingale: tendency to decrease.)

Often, $\mathscr{F}_n = \sigma(X_0, \ldots, X_n)$, $n \in \mathbb{N}_0$, called *natural filtration*.

**Example 9.7.** a) Games as in 9.5 with natural filtration, $X = \{X_n\}_{n\in\mathbb{N}}$. Then Game 1, Game 3 are not martingales, Game 2 and Game 4(?) are martingales. Game 1 is a supermartingale.

b) (dyadic martingale)

Let $\Omega = [0, 1]$ with Lebesgue measure, $f \in L^1[0, 1]$. Let

$$D_{n,k} = \Big[\frac{k}{2^n}, \frac{k+1}{2^n}\Big], \quad n \in \mathbb{N}_0, \; k = 0, 1, \ldots, 2^n - 1.$$

be the dyadic interval, let $\mathscr{F}_n$ be the $\sigma$-algebra generated by dyadic intervals of level $\leq n$, and let

$$X_n(\omega) = \sum_{k=0}^{2^n-1} \chi_{D_{n,k}}(\omega) \cdot 2^n \int_{D_{n,k}} f(\omega) d\omega, \qquad n \in \mathbb{N}_0.$$

Then $X = \{X_n\}_{n\in\mathbb{N}_0}$ is a martingale.

(i) $X_n$ is $\mathscr{F}_n$-measurable.

(ii) $\mathbb{E}[X_{n+1}|\mathscr{F}_n] = \sum_{k=0}^{2^n-1} \chi_{D_{n,k}}(\omega)\cdot 2^n \int_{D_{n,k}} X_{n+1}(\omega) d\omega = \sum_{k=0}^{2^n-1} \chi_{D_{n,k}}(\omega)\cdot 2^n \int_{D_{n,k}} f(\omega) d\omega = X_n$.

Note that $X_n(\omega) \to f(\omega)$ as $n \to \infty$ for a.e. $\omega$. This is an instance of martingale convergence theorem!

c) (Brownian motion)

Let $B_t$ be a Brownian motion on $\mathbb{R}$. For given $0 \leq t_0 < t_1 < \cdots < t_n < \cdots$, let $X_n = B_{t_n}$, $n \in \mathbb{N}_0$. Then $X = \{X_n\}_{n\in\mathbb{N}_0}$ (with natural filtration) is a martingale.

Note that $B_{t_{n+1}} - B_{t_n}$ is independent of $B_{t_0}, \ldots, B_{t_n}$, and $\mathbb{E}[B_t] = 0$. We have

(i) $X_n = B_{t_n}$ is $\mathscr{F}_n = \sigma(B_{t_0}, \ldots, B_{t_n})$-measurable.

(ii) $\mathbb{E}[X_{n+1}|\mathscr{F}_n] = \mathbb{E}[B_{t_{n+1}}|\mathscr{F}_n] = \mathbb{E}[B_{t_{n+1}} - B_{t_n}|\mathscr{F}_n] + B_{t_n} = \mathbb{E}[B_{t_{n+1}} - B_{t_n}] + B_{t_n} = B_{t_n} = X_n$.

**Definition 9.8. (Martingale; continuous-time case)** Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a probability space with filtration $\{\mathscr{F}_t\}_{t\geq 0}$, i.e., $\mathscr{F}_t \subseteq \mathscr{A}$ is a $\sigma$-algebra and $\mathscr{F}_s \subseteq \mathscr{F}_t$ for $s \leq t$. A *stochastic* (often extra technical conditions) *process* is called *adapted* if $X_t$ is $\mathscr{F}_t$-measurable for all $t \geq 0$. $X = \{X_t\}_{t\geq 0}$ is a martingale if

(i) $X$ is adapted and $\mathbb{E}[|X_t|] < \infty$ for all $t \geq 0$.

(ii) $\mathbb{E}[X_t|\mathscr{F}_s] = X_s$ for all $0 \leq s \leq t$. The natural filtration: $\mathscr{F}_t = \sigma(X_s : 0 \leq s \leq t)$.

**Example 9.9.** a) Brownian motion $\{B_t\}_{t\geq 0}$ with natural filtration is a martingale.

b) $B_t$ is Brownian motion, $\mathscr{F}_t = \sigma(B_s : 0 \leq s \leq t)$. Then $X_t = B_t^2 - t$ is a martingale.

(i) $X_t$ is adapted, and $\mathbb{E}[|X_t|] < \infty$.

(ii) $\mathbb{E}[X_t|\mathscr{F}_s] = \mathbb{E}[B_t^2 - t|\mathscr{F}_s] = \mathbb{E}[(B_t - B_s + B_s)^2|\mathscr{F}_s] - t = \mathbb{E}[(B_t - B_s)^2 + 2B_s(B_t - B_s) + B_s^2|\mathscr{F}_s] - t = \mathbb{E}[(B_t - B_s)^2|\mathscr{F}_s] + 2B_s\mathbb{E}[B_t - B_s|\mathscr{F}_s] + B_s^2 - t = \mathbb{E}[(B_t - B_s)^2] + 2B_s\mathbb{E}[B_t - B_s] + B_s^2 - t = (t - s) + B_s^2 - t = B_s^2 - s = X_s$.

Conversely,

**Theorem 9.10. (Lévy)** *Let $\{X_t\}_{t\geq 0}$ be a continuous martingale (i.e., martingale with almost surely continuous sample paths). If $X_t^2 - t$ is a martingale (w.r.t. $\mathscr{F}_t = \sigma(X_s : 0 \leq s \leq t)$), then $\{X_t\}_{t\geq 0}$ is a Brownian motion.*

Important facts about martingales: martingale convergence theorem; Doob's $L^p$-submartingale inequalities; sub- and supermartingale decompositions; optional stopping; stochastic integrals.