Homework 7 (Due: We, 6/4)

Problem 1:

(i) Prove the following version of the maximum principle: Let $B = B(z_0, r) \subseteq \mathbb{C}$ be an open disk and u be a subharmonic function on $\Omega = \mathbb{C} \setminus \overline{B}$. Suppose that there are constants $m, M \in \mathbb{R}$ such that $u \leq M$ and

$$\limsup_{z \in \Omega \to w} u(z) \leq m$$

for all $w \in \partial B$. Show that then $u \leq m$.

(ii) Show that every subharmonic function on $\mathbb{C}$ that is bounded from above is constant.

Problem 2: Show that every harmonic function u on a simply connected Riemann surface as a harmonic conjugate function v (i.e, a function such that $f = u + iv$ is holomorphic on X). Hint: Use the general theorem established in class.

Problem 3: Show that points are *removable singularities* for bounded harmonic functions: if $U \subseteq \mathbb{C}$ is open, $p \in U$, and u is a bounded harmonic function on $U \setminus \{p\}$, then u extends to a harmonic function on U.

Problem 4: Let $U, V \subseteq \mathbb{C}$ be regions and $f \colon U \to V$ be a proper holomorphic map. Show that f attains each value equally often; more precisely, show that there exists $N \in \mathbb{N}$ such that

$$N = \sum_{z \in f^{-1}(w)} \deg_f(z)$$

for each $w \in V$.