**Homework 5** (Due: Fr, 5/16)

**Problem 1:** Prove that harmonic measure has no atoms. More precisely, show that if $\Omega \subseteq \mathbb{C}$ is an open and bounded set, $z \in \Omega$ and $a \in \partial \Omega$, then $\omega_z(\{a\}; \Omega) = 0$.

**Problem 2:** Let $X$ be a compact metric space, and $V$ be a vector space of bounded functions on $X$. Suppose that $V$ contains all continuous functions and is closed under monotone limits, i.e., if $f_n \in V$ and $f_n \leq f_{n+1}$ for $n \in \mathbb{N}$, and if the pointwise limit $f = \lim_{n \to \infty} f_n$ is bounded, then $f \in V$.

Prove that then $V$ contains all bounded Borel functions on $X$. Hint: First show that the characteristic function of each open set belongs to $V$. Use this and Dynkin’s $\pi$-$\lambda$-Theorem to conclude that the characteristic function of each Borel set is in $V$.

**Problem 3:**

(i) Let $U$ and $V$ be open and bounded sets in $\mathbb{C}$, and suppose that there exists a homeomorphism $\varphi: U \to V$ such that $\varphi|_U$ is a conformal map of $U$ onto $V$. Show that if $f: \partial V \to \mathbb{R}$ is resolutive (for the generalized Dirichlet problem on $V$), then $f \circ \varphi|_U$ is resolutive (for the generalized Dirichlet problem on $U$) and we have

$$H_{f \circ \varphi|_U} = H_f \circ \varphi|_U.$$

(ii) Let $\varphi$ be a Möbius transformation with $\varphi(\mathbb{D}) = \mathbb{D}$. Show that if $z \in \mathbb{D}$ and $A \subseteq \partial \mathbb{D}$ is a Borel set, then

$$\omega_z(A; \mathbb{D}) = \omega_{\varphi(z)}(\varphi(A); \mathbb{D}).$$

(iii) Let $z \in \mathbb{D}$ and $A$ be an arc on $\partial \mathbb{D}$. Let $\alpha$ be the angle at $z$ of the circular arc triangle formed by $A$ and the hyperbolic rays from $z$ to the endpoints of $A$. Show that

$$\omega_z(A; \mathbb{D}) = \frac{\alpha}{2\pi}.$$

**Problem 4:** The purpose of this problem is to define harmonic measure for some open subsets of $\hat{\mathbb{C}}$ that are not necessarily bounded subsets of $\mathbb{C}$.

Let $U \subseteq \hat{\mathbb{C}}$ be an arbitrary open set such that $\hat{\mathbb{C}} \setminus U$ has non-empty interior. Then there exists a Möbius transformations $\varphi$ that maps $U$ onto an open and bounded set $\tilde{U}$ in $\mathbb{C}$ (why?).
(i) For \( z \in U \) we define harmonic measure of a Borel set \( A \subseteq \partial U \) as
\[
\omega_z(A; U) := \omega_{\varphi(z)}(\varphi(A); \bar{U}).
\]
Show that this defines a measure that does not depend on the choice of \( \varphi \).

(ii) Let \( \mathbb{H} = \{ z \in \mathbb{C} : \text{Im} \ z > 0 \} \) be the upper half-plane. Find an explicit formula (in terms of an integral over \( \mathbb{R} \)) for the harmonic measure \( \omega_z(A; \mathbb{H}) \) of a Borel set \( A \subseteq \mathbb{R} \subseteq \partial \mathbb{H} \) with respect to a point \( z \in \mathbb{H} \).