Homework 5 (Due: Fr, 5/16)

Problem 1: Prove that harmonic measure has no atoms. More precisely, show that if $\Omega \subseteq \mathbb{C}$ is an open and bounded set, $z \in \Omega$ and $a \in \partial\Omega$, then $\omega_z(\{a\}; \Omega) = 0$.

Problem 2: Let X be a compact metric space, and V be a vector space of bounded functions on X. Suppose that V contains all continuous functions and is closed under monotone limits, i.e., if $f_n \in V$ and $f_n \leq f_{n+1}$ for $n \in \mathbb{N}$, and if the pointwise limit $f = \lim_{n\to\infty} f_n$ is bounded, then $f \in V$.

Prove that then V contains all bounded Borel functions on X. Hint: First show that the characteristic function of each open set belongs to V. Use this and Dynkin's π-λ-Theorem to conclude that the characteristic function of each Borel set is in V.

Problem 3:

(i) Let U and V be open and bounded sets in $\mathbb{C}$, and suppose that there exists a homeomorphism $\varphi\colon \overline{U} \to \overline{V}$ such that $\varphi|_U$ is a conformal map of U onto V. Show that if $f\colon \partial V \to \mathbb{R}$ is resolutive (for the generalized Dirichlet problem on V), then $f \circ \varphi|_{\partial U}$ is resolutive (for the generalized Dirichlet problem on U) and we have

$$H_f \circ \varphi|_U = H_{f\circ\varphi|_{\partial U}}.$$

(ii) Let φ be a Möbius transformation with $\varphi(\mathbb{D}) = \mathbb{D}$. Show that if $z \in \mathbb{D}$ and $A \subseteq \partial\mathbb{D}$ is a Borel set, then

$$\omega_z(A; \mathbb{D}) = \omega_{\varphi(z)}(\varphi(A); \mathbb{D}).$$

(iii) Let $z \in \mathbb{D}$ and A be an arc on $\partial\mathbb{D}$. Let α be the angle at z of the circular arc triangle formed by A and the hyperbolic rays from z to the endpoints of A. Show that

$$\omega_z(A; \mathbb{D}) = \frac{\alpha}{2\pi}.$$

Problem 4: The purpose of this problem is to define harmonic measure for some open subsets of $\widehat{\mathbb{C}}$ that are not necessarily bounded subsets of $\mathbb{C}$.

Let $U \subseteq \widehat{\mathbb{C}}$ be an arbitrary open set such that $\widehat{\mathbb{C}} \setminus U$ has non-empty interior. Then there exists a Möbius transformations φ that maps U onto an open and bounded set $\widetilde{U}$ in $\mathbb{C}$ (why?).

(i) For $z \in U$ we define harmonic measure of a Borel set $A \subseteq \partial U$ as

$$\omega_z(A; U) := \omega_{\varphi(z)}(\varphi(A); \widetilde{U}).$$

Show that this defines a measure that does not depend on the choice of φ.

(ii) Let $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ be the upper half-plane. Find an explicit formula (in terms of an integral over $\mathbb{R}$) for the harmonic measure $\omega_z(A; \mathbb{H})$ of a Borel set $A \subseteq \mathbb{R} \subseteq \partial\mathbb{H}$ with respect to a point $z \in \mathbb{H}$.