Homework 4 (Due: Fr, 5/9)

Problem 1: Let $\Omega \subseteq \mathbb{C}$ be a bounded and open set, and u be a subharmonic function on Ω that is bounded from above. Suppose that there exists a finite set $F \subseteq \partial\Omega$ such that

$$\limsup_{z \to z_0} u(z) \leq 0$$

for all $z_0 \in \partial\Omega \setminus F$. Show that then $u \leq 0$.

Problem 2: Let $\Omega = \mathbb{D} \setminus \{0\}$, and define $f \colon \partial\Omega = \partial\mathbb{D} \cup \{0\} \to \mathbb{R}$ by setting $f(z) = 0$ for $z \in \partial\mathbb{D}$ and $f(0) = 1$. Show that then $\underline{H}_f = \overline{H}_f \equiv 0$.

Problem 3: Let $f \colon \partial\mathbb{D} \to \mathbb{R}$ be an integrable function (w.r.t. Lebesgue measure on $\partial\mathbb{D}$), and

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|z - e^{it}|^2} f(e^{it})\, dt, \quad z \in \mathbb{D},$$

be its Poisson extension.

Show that then

$$\liminf_{\zeta \in \partial\mathbb{D} \to \zeta_0} f(\zeta) \leq \liminf_{z \in \mathbb{D} \to \zeta_0} u(z) \leq \limsup_{z \in \mathbb{D} \to \zeta_0} u(z) \leq \limsup_{\zeta \in \partial\mathbb{D} \to \zeta_0} f(\zeta)$$

for all $\zeta_0 \in \partial\mathbb{D}$.

Problem 4: Let $\Omega \subseteq \mathbb{C}$ be a bounded and open set, and u be subharmonic function on Ω. Show that then $u(z_0) = \limsup_{z \to z_0} u(z)$ for all $z_0 \in \Omega$.