Homework 3 (Due: Fr, 5/2)

Problem 1: Let (X, d) be a compact metric space, and $f\colon X \to [-\infty, \infty)$ be an upper semicontinuous function with $f \not\equiv -\infty$. For $n \in \mathbb{N}$ and $x \in X$ define

$$u_n(x) = \sup_{y \in X,\, f(y) \neq -\infty} \{f(y) + 1/n - nd(x, y)\}.$$

Show that then u_n is a continuous function on X with values in $\mathbb{R}$, that $f \leq u_{n+1} \leq u_n$ for $n \in \mathbb{N}$ and that

$$\lim_{n\to\infty} u_n(x) = f(x)$$

for all $x \in X$.

Problem 2: Let $U \subseteq \mathbb{C}$ be open and u be a locally integrable function on U. We pick a function $\varphi \in C_c^\infty(\mathbb{C})$ with $\varphi \geq 0$, $\operatorname{supp}(\varphi) \subseteq B(0, 1)$, and $\int_{\mathbb{C}} \varphi\, dA = 1$ such that φ is *radial* (i.e., $\varphi(z) = \varphi(|z|)$ for $z \in \mathbb{C}$). For $\epsilon > 0$ define

$$\varphi_\epsilon(z) := \frac{1}{\epsilon^2}\varphi(z/\epsilon)$$

for $z \in \mathbb{C}$ and $U_\epsilon := \{z \in U : B(z, \epsilon) \subseteq U\}$.

a) Let $u_\epsilon = u * \varphi_\epsilon$ on U_ϵ. Show that there exist a subharmonic function v on U with $u = v$ almost everywhere on U if and only if u_ϵ is subharmonic on U_ϵ for each $\epsilon > 0$.

b) Show that there exist a subharmonic function v on U with $u = v$ almost everywhere on U if if and only if $\int_U u\Delta\psi\, dA \geq 0$ for all $\psi \in C_c^\infty(U)$ with $\psi \geq 0$.

c) Show that if u is subharmonic, then there exists a Borel measure $\mu \geq 0$ on U such that

$$\int_U u\Delta\psi\, dA = \int_U \psi\, d\mu$$

for all $\psi \in C_c^\infty(U)$.

Problem 3: Let $U \subseteq \mathbb{C}$ be open, and u be a continuous real-valued function on U. Show that u is harmonic if and only if $\int_U u\Delta\psi\, dA = 0$ for all $\psi \in C_c^\infty(U)$.

Problem 4: Let $U, V \subseteq \mathbb{C}$ be regions, $f\colon V \to U$ be a non-constant holomorphic map, and u be subharmonic on U. Show that $v = u \circ f$ is subharmonic on V.