Homework 2 (Due: Fr, 4/25)

Problem 1: Let $p > 0$, $R > 0$, $U \subseteq \mathbb{C}$ be an open set, and $a \in U$ such that $B(a, R) \subseteq U$. Show that if f is a holomorphic function on U, then the function

$$r \mapsto \int_0^{2\pi} |f(re^{it})|^p \, dt$$

is non-decreasing on the interval $[0, R)$.

Problem 2: Consider the subharmonic function $u(z) = \log |z|$ on $\mathbb{C}$. Find the distributional Laplacian μ of u, i.e., find a Borel measure μ on $\mathbb{C}$ such that

$$\int_{\mathbb{C}} u \, \Delta\varphi \, dA = \int_{\mathbb{C}} \varphi \, d\mu$$

for all $\varphi \in C_c^\infty(\mathbb{C})$.

Problem 3: Let u be a C^2-smooth subharmonic function on $\mathbb{C}$.

a) Show that

$$\int_{\mathbb{C}} \nabla u \cdot \nabla \varphi \, dA \leq 0$$

for all $\varphi \in C_c^1(\mathbb{C})$ with $\varphi \geq 0$.

b) Suppose in addition that $u \geq 0$. Show that then there exists a constant $C > 0$ such that u satisfies the *Caccioppoli inequality*

$$\int_{\mathbb{C}} |\nabla u|^2 \psi^2 \, dA \leq C \int_{\mathbb{C}} u^2 |\nabla \psi|^2 \, dA$$

for all $\psi \in C_c^\infty(\mathbb{C})$ with $\psi \geq 0$. Hint: Make a clever choice of the function φ in (a).

c) Use (b) to give another proof of the fact that every bounded harmonic function on $\mathbb{C}$ is constant.

Problem 4: We denote by $H = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ the upper half-plane.

a) Let $I = [a, b] \subseteq \mathbb{R}$ be a fixed interval. For a point $p \in U$, let $\theta(p) \in (0, \pi)$ be the angle under which I is seen from p, i.e., $\theta(p)$ is the angle between the line segments $[p, a]$ and $[p, b]$. Show that the function $p \mapsto \theta(p)$ is harmonic on H.

b) Let $u\colon \overline{H} \to \mathbb{R}$ be a continuous function. Suppose that $u|H$ is harmonic and that $u|\mathbb{R} \leq 0$. Show that if

$$|u(z)| = o(|z|)$$

as $|z| \to \infty$ (i.e., $\lim_{|z|\to\infty} |u(z)|/|z| = 0$), then $u \leq 0$. Hint: Use (a) to find a suitable comparison function on a large half-disk.