Homework 1 (Due: We, 4/9)

Problem 1: Let $U \subseteq \mathbb{C}$ be an open set, and f and g be locally integrable functions on U. The function g is called the *weak* or *distributional* $\bar{\partial}$-derivative of f if

$$\int_U g\varphi \, dA = -\int_U f\varphi_{\bar{z}} \, dA$$

for all $\varphi \in C_c^\infty(U)$.

a) Show that if the weak $\bar{\partial}$-derivative g of f exists, then it is unique in the following sense: if $\tilde{g}$ is another weak $\bar{\partial}$-derivative of f, then $g = \tilde{g}$ a.e. (=almost everywhere) on U. Hint: You can use results from measure theory about the density of relevant function spaces in the space of integrable functions without proof.

b) Suppose that f is C^1-smooth on U. Show that then the $\bar{\partial}$-derivative $f_{\bar{z}}$ in the usual sense is the weak $\bar{\partial}$-derivative of f.

c) Define the *weak gradient* of a locally integrable function $f\colon U \to \mathbb{R}$, formulate statements analogous to (a) and (b), and give an indication for the proofs.

Problem 2: The purpose of this problem is to prove (the holomorphic version of) *Weyl's Lemma*: Let $U \subseteq \mathbb{C}$ be open, and $f\colon U \to \mathbb{C}$ be a locally integrable function. Suppose that the weak $\bar{\partial}$-derivative $f_{\bar{z}}$ of f exists, and that $f_{\bar{z}} = 0$ a.e. on U. Then there exists a holomorphic function $\tilde{f}$ on U such that $\tilde{f} = f$ a.e. on U.

For the proof pick a function $\varphi \in C_c(\mathbb{C})$ with $\operatorname{supp}(\varphi) \subseteq B(0,1)$, $\int_{\mathbb{C}} \varphi \, dA = 1$ and $\varphi(-z) = \varphi(z)$ for all $z \in \mathbb{C}$. For $\epsilon > 0$ define

$$\varphi_\epsilon(z) := \frac{1}{\epsilon^2}\varphi(z/\epsilon)$$

for $z \in \mathbb{C}$, and $U_\epsilon := \{z \in U : B(z,\epsilon) \subseteq U\}$. Now complete the following steps:

a) If $g \in C_c^\infty(U_\epsilon)$ and h is a locally integrable on U, then

$$\int_{U_\epsilon} g \cdot (\varphi_\epsilon * h) \, dA = \int_U h \cdot (\varphi_\epsilon * g) \, dA$$

(note that $\varphi_\epsilon * g$ is defined on U if we consider g as a function on $\mathbb{C}$ by setting it $\equiv 0$ outside U_ϵ). Carefully justify the existence of all integrals!

b) Let h be a locally integrable function on U, and suppose h has the weak $\bar{\partial}$-derivative k on U. We know that $h * \varphi_\epsilon$ is smooth on U_ϵ (why?). Show that $(h * \varphi_\epsilon)_{\bar{z}} = k * \varphi_\epsilon$ on U_ϵ.

Now let f be as in Weyl's Lemma, i.e., f is locally integrable on U and has a weak $\bar{\partial}$-derivative that vanishes almost everywhere on U.

c) Show that then $f * \varphi_\epsilon$ is holomorphic on U_ϵ.

d) Show that there is a sequence $\{\epsilon_n\}$ of numbers $\epsilon_n > 0$ with $\lim_{n\to\infty} \epsilon_n = 0$ such that the sequence $\{f_n\}$ of functions defined by $f_n := f * \varphi_{\epsilon_n}$ converges locally uniformly on U (note that for each sufficiently small neighborhood N of a point in U the function f_n is defined on N for all large n; so the locally uniform convergence of $\{f_n\}$ on U can be given a meaningful interpretation).

e) Prove Weyl's Lemma by finding a holomorphic function $\tilde{f}$ on U such that $\tilde{f} = f$ a.e. on U.

Problem 3: Let $U, V \subseteq \mathbb{C}$ be regions, and $f\colon U \to V$ be a map. Show that f is holomorphic or *anti-holomorphic* (i.e., its complex conjugate $\bar{f}$ holomorphic) if and only if $h \circ f$ is harmonic for each harmonic function $h\colon V \to \mathbb{R}$.

Problem 4: Let $U \subseteq \mathbb{C}$ be open, and $f \in C(U)$.

a) Suppose that there exists a sequence $\{f_n\}$ of holomorphic function on U such that
$$\lim_{n\to\infty} \int_U f_n \varphi \, dA = \int_U f \varphi \, dA$$
for all $\varphi \in C_c^\infty(U)$. Show that then f is holomorphic on U.

b) State and prove a version of (a), where f is only assumed to be locally integrable on U.