**Problem 1:** Let $U \subseteq \mathbb{C}$ be an open set, and $f$ and $g$ be locally integrable functions on $U$. The function $g$ is called the weak or distributional $\bar{\partial}$-derivative of $f$ if
\[
\int_U g \varphi \, dA = -\int_U f \varphi \, dA
\]
for all $\varphi \in C_c^\infty(U)$.

a) Show that if the weak $\bar{\partial}$-derivative $g$ of $f$ exists, then it is unique in the following sense: if $\tilde{g}$ is another weak $\bar{\partial}$-derivative of $f$, then $g = \tilde{g}$ a.e. (=almost everywhere) on $U$. Hint: You can use results from measure theory about the density of relevant function spaces in the space of integrable functions without proof.

b) Suppose that $f$ is $C^1$-smooth on $U$. Show that then the $\bar{\partial}$-derivative $f_{\bar{z}}$ in the usual sense is the weak $\bar{\partial}$-derivative of $f$.

c) Define the weak gradient of a locally integrable function $f : U \to \mathbb{R}$, formulate statements analogous to (a) and (b), and give an indication for the proofs.

**Problem 2:** The purpose of this problem is to prove (the holomorphic version of) Weyl’s Lemma: Let $U \subseteq \mathbb{C}$ be open, and $f : U \to \mathbb{C}$ be a locally integrable function. Suppose that the weak $\bar{\partial}$-derivative $f_{\bar{z}}$ of $f$ exists, and that $f_{\bar{z}} = 0$ a.e. on $U$. Then there exists a holomorphic function $\tilde{f}$ on $U$ such that $\tilde{f} = f$ a.e. on $U$.

For the proof pick a function $\varphi \in C_c^\infty(\mathbb{C})$ with $\text{supp}(\varphi) \subseteq B(0, 1)$, $\int_{\mathbb{C}} \varphi \, dA = 1$ and $\varphi(-z) = \varphi(z)$ for all $z \in \mathbb{C}$. For $\epsilon > 0$ define
\[
\varphi_\epsilon(z) := \frac{1}{\epsilon^2} \varphi(z/\epsilon)
\]
for $z \in \mathbb{C}$, and $U_\epsilon := \{ z \in U : B(z, \epsilon) \subseteq U \}$. Now complete the following steps:

a) If $g \in C_c^\infty(U_\epsilon)$ and $h$ is a locally integrable on $U$, then
\[
\int_{U_\epsilon} g \cdot (\varphi_\epsilon \ast h) \, dA = \int_U h \cdot (\varphi_\epsilon \ast g) \, dA
\]
(note that $\varphi_\epsilon \ast g$ is defined on $U$ if we consider $g$ as a function on $\mathbb{C}$ by setting it $\equiv 0$ outside $U_\epsilon$). Carefully justify the existence of all integrals!

b) Let $h$ be a locally integrable function on $U$, and suppose $h$ has the weak $\bar{\partial}$-derivative $k$ on $U$. We know that $h \ast \varphi_\epsilon$ is smooth on $U_\epsilon$ (why?). Show that $(h \ast \varphi_\epsilon)_{\bar{z}} = k \ast \varphi_\epsilon$ on $U_\epsilon$. 

Now let $f$ be as in Weyl’s Lemma, i.e., $f$ is locally integrable on $U$ and has a weak $\bar{\partial}$-derivative that vanishes almost everywhere on $U$.

c) Show that then $f \ast \varphi_{\epsilon}$ is holomorphic on $U_{\epsilon}$.

d) Show that there is a sequence $\{\epsilon_n\}$ of numbers $\epsilon_n > 0$ with $\lim_{n \to \infty} \epsilon_n = 0$ such that the sequence $\{f_n\}$ of functions defined by $f_n := f \ast \varphi_{\epsilon_n}$ converges locally uniformly on $U$ (note that for each sufficiently small neighborhood $N$ of a point in $U$ the function $f_n$ is defined on $N$ for all large $n$; so the locally uniform convergence of $\{f_n\}$ on $U$ can be given a meaningful interpretation).

e) Prove Weyl’s Lemma by finding a holomorphic function $\tilde{f}$ on $U$ such that $\tilde{f} = f$ a.e. on $U$.

**Problem 3:** Let $U, V \subseteq \mathbb{C}$ be regions, and $f : U \to V$ be a map. Show that $f$ is holomorphic or anti-holomorphic (i.e., its complex conjugate $\bar{f}$ holomorphic) if and only if $h \circ f$ is harmonic for each harmonic function $h : V \to \mathbb{R}$.

**Problem 4:** Let $U \subseteq \mathbb{C}$ be open, and $f \in C(U)$.

a) Suppose that there exists a sequence $\{f_n\}$ of holomorphic function on $U$ such that

$$\lim_{n \to \infty} \int_U f_n \varphi \, dA = \int_U f \varphi \, dA$$

for all $\varphi \in C^\infty_c(U)$. Show that then $f$ is holomorphic on $U$.

b) State and prove a version of (a), where $f$ is only assumed to be locally integrable on $U$. 