Final Exam (Due: Mo, 6/9, 2pm)

There are four problems with a total of 100 pts.

Problem 1: Here we consider Green's functions for bounded regions in $\mathbb{C}$ defined as minimal elements in certain classes of superharmonic functions. On Riemann surfaces we defined Green's functions as pointwise supremums of certain classes of subharmonic functions. The purpose of this problem is to reconcile these two notions.

(a) Let U and V be bounded regions in $\mathbb{C}$, and G_U and G_V be their respective Green's functions. Show that if $U \subseteq V$, then $G_U(z,w) \leq G_V(z,w)$ for all $z, w \in U$. (5 pts)

(b) Let U and U_n for $n \in \mathbb{N}$ be bounded regions in $\mathbb{C}$, and G and G_n be their respective Green's functions. Suppose that $U_n \subseteq U_{n+1}$ for $n \in \mathbb{N}$, and $U = \bigcup_{n\in\mathbb{N}} U_n$. Show that then

$$G(z,w) = \lim_{n\to\infty} G_n(z,w)$$

for all $z, w \in U$ (note that $G_n(z,w)$ is defined for sufficently large n). Hint: Fix $w \in U$, and consider the pointwise limit of the right hand side in the asserted equality as a function of z. (5 pts)

(c) Let U be a bounded region in $\mathbb{C}$. Outline an argument for the existence of a sequence of regions U_n, $n \in \mathbb{N}$, with the following properties: U_n is compactly contained in U, each boundary point of U_n is regular for the Dirichlet problem, $U_n \subseteq U_{n+1}$ for $n \in \mathbb{N}$, and $\bigcup_{n\in\mathbb{N}} U_n = U$. (5 pts)

(d) Let V be a bounded region in $\mathbb{C}$ such that every boundary point of V is regular for the Dirichlet problem, and G_V be its Green's function. Show that if $w \in V$ and we define a function $v\colon \mathbb{C} \setminus \{w\} \to \mathbb{R}$ by setting $v(z) = G(z,w)$ for $z \in V \setminus \{w\}$ and $v(z) = 0$ for $z \in \mathbb{C} \setminus V$, then v is continuous subharmonic function on $\mathbb{C} \setminus \{w\}$. (5 pts)

(e) Let U be a bounded region in $\mathbb{C}$ and G_U be its Green's function. Fix $w \in U$ and let $\mathcal{F}$ be the set of all continuous subharmonic function v on $U \setminus \{w\}$ with the following properties:

(i) $\limsup_{z\to w}(v(z) + \log|z-w|) < +\infty$,

(ii) there exists a compact set K with $w \in K \subseteq U$ such that $v(z) = 0$ for all $z \in U \setminus K$.

Show that then
$$G_U(z,w) = \sup_{v\in\mathcal{F}} v(z)$$
for all $z \in U \setminus \{w\}$. (5 pts)

Problem 2: The purpose of this problem is to show that if a Riemann surface Y can be represented as $Y = X \setminus N$, where X is another Riemann surface and $N \subseteq X$ is a closed set with non-empty interior, then the Green's function $G_Y(\cdot, p_0)$ on Y exists for each $p_0 \in Y$.

For the construction we pick a coordinate disk D whose closure is contained in the interior of N (this is possible by the hypotheses). Let $p_0 \in Y$ be arbitrary. We pick another coordinate disk D_0 and a chart z on X defined in a neighborhood of $\overline{D}_0$ such that $p_0 \in D_0$, $z(p_0) = 0$ and $\overline{D}_0 \cap \overline{D} = \emptyset$. Set $D_0' = z^{-1}(B(0,1/2))$. Note that then $\overline{D}_0' \subseteq D_0$.

(a) Consider the class $\mathcal{F}$ of all continuous and subharmonic functions on $Z := X \setminus (\overline{D} \cup \overline{D}_0')$ such that for each $v \in \mathcal{F}$ there exists a compact set $K \subseteq X$ such that $v|(Z \setminus K) \equiv 0$, and
$$\limsup_{p\to q} v(p) \le \begin{cases} 0 & \text{for } q \in \partial D, \\ 1 & \text{for } q \in \partial D_0'. \end{cases}$$
Let $u = \sup_{v\in\mathcal{F}} v$ be the pointwise supremum on Z. Show that u is a harmonic function on Z and that u has a continuous extension to $\overline{Z} = X \setminus (D \cup D_0')$ such that $u|\partial D = 0$ and $u|\partial D_0' = 1$ (for the last part it is enough to give a sketch of the argument). (5 pts)

(b) Show that $0 < u < 1$ on Z and that there exists $\delta > 0$ such that $u|\partial D_0 \le 1 - \delta$. (5 pts)

(c) Pick $M \ge \log 2$ such that $M\delta > \log 2$, where δ as in (b). Define a function w on $X \setminus (\overline{D} \cup \{p_0\})$ by setting
$$w(p) := \begin{cases} L(p) := \log \frac{1}{|z(p)|} + M - \log 2 & \text{for } p \in D_0' \setminus \{p_0\}, \\ \min\{L(p), Mu(p)\} & \text{for } p \in D_0 \setminus D_0', \\ Mu(p) & \text{for } p \in X \setminus (D_0 \cup \overline{D}). \end{cases}$$
Show that w is superharmonic on $X \setminus (\overline{D} \cup \{p_0\})$. (10 pts)

(d) Show that the Green's function $G_Y(\cdot, p_0)$ exists. (5 pts)

Problem 3: Let $\mathcal{S}$ be the family of all conformal maps $f\colon \mathbb{D} \to f(\mathbb{D}) \subseteq \mathbb{C}$ satisfying $f(0) = f'(0) - 1 = 0$. The purpose of this problem is to show that $\mathcal{S}$ is a normal family (this easily follows from the *Koebe distortion theorem*, but your arguments should not rely on this).

(a) Show that if $f \in \mathcal{S}$, and $d_f := \inf\{|w| : w \in \mathbb{C} \setminus f(\mathbb{D})\}$, then the infimum is attained as a minimum and $0 < d_f \leq 1$. (4 pts)

(b) For $f \in \mathcal{S}$ consider the auxiliary function $g_f := f/w_f - 1$, where $w_f \in \mathbb{C} \setminus f(\mathbb{D})$ satisfies $|w_f| = d_f$. Then g_f has a holomorphic square root denoted by h_f, and by a suitable choice of the branch we may assume $h_f(0) = i$ (justify these statements!). Show that there exists $r > 0$ independent of f such that $B(-i, r) \cap h_f(\mathbb{D}) = \emptyset$. (7 pts)

(c) Show that $\mathcal{S}$ is a normal family. Hint: It suffices to show that $\mathcal{S}$ is locally uniformly bounded. Argue by contradiction, and suppose $\{f_n\}$ is a sequence in $\mathcal{S}$ and $\{z_n\}$ is a sequence of points in some compact set $K \subseteq \mathbb{D}$ such that $|f_n(z_n)| \geq n$ for all $n \in \mathbb{N}$. Now consider the functions $k_n := 1/(h_n + i)$, where $h_n := h_{f_n}$ for $n \in \mathbb{N}$ is defined as in (b). (7 pts)

(d) The previous argument actually shows that the family $\{g_f : f \in \mathcal{S}\}$ of the auxiliary maps as defined in (b) is locally uniformly bounded (why?). Use this to show that there exists $\rho_0 > 0$ such that $B(0, \rho_0) \subseteq f(\mathbb{D})$ for *all* $f \in \mathcal{S}$.

(The largest possible value of ρ_0 in the last statement is known as the *Koebe radius*. One can show that $\rho_0 = 1/4$.) (7 pts)

Problem 4: The purpose of this problem is to outline the proof of the Uniformization Theorem in the general case.

(a) Show that if Ω is a simply connected region in a Riemann surface X such that $X \setminus \Omega$ has non-empty interior, then Ω is biholomorphic to $\mathbb{D}$. (2 pts)

Now let X be an arbitrary *simply connected* Riemann surface. Then one can find simply connected regions $\Omega_n \subseteq X$ for $n \in \mathbb{N}_0$ such that $\overline{\Omega}_n \subseteq \Omega_{n+1}$ for $n \in \mathbb{N}_0$ and $X' := \bigcup_{n\in\mathbb{N}_0} \Omega_n = X$ if X is open (i.e., non-compact), and $X' = \bigcup_{n\in\mathbb{N}_0} \Omega = X \setminus \{p\}$ for a suitable point $p \in X$ if X is compact.

The existence of such an exhaustion is obvious if one relies on the topological classification of surfaces (it implies that X is *homeomorphic* to $\mathbb{C}$ or $\widehat{\mathbb{C}}$), but non-trivial to prove from scratch. In the following we will assume the existence of the exhaustion.

(b) Pick a basepoint $p_0 \in \Omega_0$. Show that for $n \in \mathbb{N}_0$ there exists a biholomorphism φ_n of Ω_n onto a Euclidean disk $B(0, r_n)$ of radius $r_n > 0$ such that $\varphi_n(p_0) = 0$ and $(\varphi_n \circ \varphi_0^{-1})'(0) = 1$ for $n \in \mathbb{N}_0$. (3 pts)

(c) Show that $r_{n+1} > r_n$ for $n \in \mathbb{N}_0$. (2 pts)

Let $R = \sup_{n\in\mathbb{N}_0} r_n \in (0, +\infty]$.

(d) Show that if $R < +\infty$, then the sequence $\{\varphi_n\}$ subconverges (in a suitable sense) to a biholomorphic map $\varphi\colon X' \to B(0, R)$ and $X' = X$. (7 pts)

(e) Show that if $R = +\infty$, then the sequence $\{\varphi_n\}$ subconverges to a biholomorphic map $\varphi\colon X' \to \mathbb{C}$. (7 pts)

(f) Show that X is biholomorphic to $\mathbb{D}$, $\mathbb{C}$, or $\widehat{\mathbb{C}}$. (4 pts)