Correction

In the proof of the uniformization theorem the following lemma is needed.

Lemma 1. *Let X be a simply connected Riemann surface and suppose that for some $p_0 \in X$ the Green's function $G = G(\cdot, p_0)$ exists. Then there is a holomorphic function $F\colon X \to \mathbb{C}$ such that $|F| = e^{-G}$.*

My intended proof used a theorem discussed in class, but relied on the following incorrect reasoning: if f and g are non-constant holomorphic functions on a region $U \subseteq X$ with $|f| = |g|$ and there exists a point $p \in U$ with $f(p) = g(p)$, then $f = g$. This is actually true if $f(p) = g(p) \neq 0$, but not if $f(p) = g(p) = 0$.

Unfortunately, there is no entirely straightforward resolution of the problem. To get a correct proof, one needs a modification of the theorem we proved in class.

To formulate a suitable modification, we first need a definition: we say that two functions f and g defined in a neighborhood of a point *define the same germ at* p if for a sufficiently small neighborhood W of p we have $f|W = g|W$. In the modified theorem and its proof we replace all statements of the form "$f(p) = g(p)$" with the statement "f and g define the same germ at p". Here is the precise version of the modified theorem and an outline for its proof.

Theorem 1. *Let X be a simply connected Riemann surface, and suppose that for each open set $U \subseteq X$ we are given a (possibly empty) family $\mathcal{F}(U)$ of functions on U with the following properties:*

- (A1) *If $U, V \subseteq X$ are open sets with $U \subseteq V$, and $f \in \mathcal{F}(V)$, then $f|U \in \mathcal{F}(U)$.*
- (A2) *If $U \subseteq X$ is a region, $f, g \in \mathcal{F}(U)$, and f and g define the same germ at a point $p \in U$, then $f = g$.*
- (A3) *For every point $p \in U$ there exists a region $U \subseteq X$ with $p \in U$ such that $\mathcal{F}(U) \neq \emptyset$ and the following statement is true: if $V \subseteq X$ is open, $g \in \mathcal{F}(V)$, and $q \in U \cap V$, then there exists a function $f \in \mathcal{F}(U)$ such that f and g define the same germ at q.*

Then there exists a function F on X such that for each point $p \in X$ there exists an open set $U \subseteq X$ with $p \in U$ and $F|U \in \mathcal{F}(U)$.

Note that axiom A2 is weaker than in the original theorem, but A3 is stronger.

Outline of proof. The proof is along the lines of what we discussed in class: one considers all pairs (p, f) of points and functions for which there exists an open set $U \subseteq X$ with $p \in U$ and $f \in \mathcal{F}(U)$. One defines a relation as follows: $(p, f) \sim (q, g)$ iff $p = q$ and f and g define the same germ at $p = q$. Then $\sim$ is an equivalence relation on the set of these pairs. We denote the equivalence class of a pair (p, f) by

$[p, f]$ and define Y as the set of all equivalence classes $[p, f]$. If we set $\pi([p, f]) := p$, then we get a well-defined map $\pi\colon Y \to X$.

If $U \subseteq X$ is open and $f \in \mathcal{F}(U)$, then we define $[U, f] := \{[p, f] : p \in U\}$. By the same argument as before, one can show that the sets $[U, f]$ form the basis of a Hausdorff topology on Y and that $\pi\colon Y \to X$ is a covering map. Since X is simply connected, there exists a continuous map $\varphi\colon X \to Y$ such that $\pi \circ \varphi$ is the identity map on X. For $p \in X$ we now set $F(p) = f(p)$ if $\varphi(p) = [p, f]$. Then F is well-defined on X and has the desired properties. $\square$

The definition of Y follows the well-known and standard construction of the *étalé space* in the theory of sheaves (see, for example, Wells, "Differential Analysis on Complex Manifolds", 2nd ed., Chap. 2, Sect. 2).

We can now prove the lemma.

Proof of the lemma. For each open set $U \subseteq X$ we let $\mathcal{F}(U)$ be the set of all holomorphic functions f on U such that $|f| = e^{-G}$. Then axioms A1–A3 in the theorem are satisfied:

A1 is obvious, and A2 is true by the uniqueness theorem for holomorphic functions (note that if f and g are as in A2, then they agree not only at the point p, but in a whole neighborhood of p).

To verify A3, let $p \in X$ be arbitrary. For U we pick a coordinate disk U containing p. If $p \neq p_0$, we choose U so small such that $p_0 \notin U$.

So if $p \neq p_0$, then $G|U$ is harmonic, and hence it has a harmonic conjugate $\tilde{G}$ on U. Then $f = \exp(-(G + i\tilde{G})) \in \mathcal{F}(U)$, and f is uniquely determined up to a rotation factor. If V, g, and q are as in A3, then by modifying f by such a factor, we can guarantee that f and g agree on a neighborhood of q, and so define the same germ there.

If $p = p_0$, the argument in the last part is the same, once we have established the *existence* of a function $f \in \mathcal{F}(U)$. For this we choose a chart z on U mapping U to the unit disk $\mathbb{D}$ such that $z(p_0) = 0$. We know that then on $U \setminus \{p_0\}$ we have $G = -\log|z| + H$, where H is harmonic on U. If we choose a harmonic conjugate $\tilde{H}$ of H on U and set $f = z\exp(-(H + i\tilde{H}))$, then f is holomorphic on U and $|f| = e^{-G}$ on U. So $f \in \mathcal{F}(U)$. Axiom A3 follows.

By the theorem there exists a function F on X such that every point in X has an open neighborhood U with $F|U \in \mathcal{F}(U)$. Then F is holomorphic on X and $|F| = e^{-G}$ as desired. $\square$