Complex Analysis

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Course notes for Math 246A and 246B

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Fall 2011 and Winter 2012

Contents

Preface

These notes cover the material of a course on complex analysis that I taught in the fall quarter of 2011 and the winter quarter of 2012. In many respects I closely follow Rudin's book on "Real and Complex Analysis". Since Walter Rudin is the unsurpassed master of mathematical exposition for whom I have great admiration, I saw no point in trying to improve on his presentation of subjects that are relevant for the course.

The notes give a fairly accurate account of the material covered in class. They are rather terse as oral discussions that gave further explanations or put results into perspective are mostly omitted. In addition, all pictures and diagrams are currently missing as it is much easier to produce them on the blackboard than to put them into print.

1 Algebraic properties of complex numbers

1.1. Intuitive idea. Complex numbers are expressions of the form $a + bi$ with real numbers a and b. Here i is the *imaginary unit*. One computes (i.e., adds and multiplies) with complex numbers as usual, but sets $i^2 = i \cdot i := -1$. For example,

$$(3 + 5i)(2 + i) = 6 + 10i + 3i + 5i^2 = 6 + 13i - 5 = 1 + 13i.$$

1.2. Definition of the complex numbers. For a rigorous definition we let the set of *complex numbers* be

$$\mathbb{C} := \{(a, b) : a, b \in \mathbb{R}\}$$

with the correspondence $(a, b) \cong a + bi$ in mind. *Addition* and *multiplication* are defined accordingly:

$$\begin{aligned} (a, b) + (c, d) &:= (a + c, b + d), \\ (a, b) \cdot (c, d) &:= (ac - bd, ad + bc). \end{aligned}$$

One often omits the multiplication sign and writes $zw := z \cdot w$ for $z, w \in \mathbb{C}$. One also uses the convention that multiplication binds stronger than addition. So $u + vw = u + (v \cdot w)$ for $u, v, w \in \mathbb{C}$, etc.

That one computes with complex numbers "as usual" is mathematically expressed by the following fact.

Theorem 1.3. $(\mathbb{C}, +, \cdot)$ *is a field, that is:*

1. $(\mathbb{C}, +)$ *is an abelian group, which means that*

   1.1 *the addition* $+$ *is associative,*

   1.2 *there exists a neutral element* $0 := (0, 0) \in \mathbb{C}$ *with respect to addition,*

   1.3 *every element* $(a, b) \in \mathbb{C}$ *has an (additive) inverse* $(-a, -b) \in \mathbb{C}$,

   1.4 *the addition* $+$ *is commutative.*

2. $(\mathbb{C}, +, \cdot)$ *is a commutative ring, which means that*

2.1 $(\mathbb{C}, +)$ *is an abelian group,*

2.2 the multiplication $\cdot$ *is associative and commutative,*

2.3 the distributive law holds.

3. $(\mathbb{C}^*, \cdot)$ *is a group, where* $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$.

The proof is straightforward, but tedious, and so we skip it.

Remark 1.4. If we write (for the moment), $\tilde{a} := (a, 0) \in \mathbb{C}$ for $a \in \mathbb{R}$, then

$$\begin{aligned} \tilde{a} + \tilde{b} &= (a,0) + (b,0) \\ &= (a+b, 0) \;=\; \widetilde{a+b}, \end{aligned}$$

and

$$\begin{aligned} \tilde{a} \cdot \tilde{b} &= (a,0) \cdot (b,0) \\ &= (ab, 0) \;=\; \widetilde{ab}. \end{aligned}$$

This means that with expressions $\tilde{a}$, $\tilde{b}$, etc., one can compute in exactly the same way as with the underlying real numbers a, b, etc. More precisely, the map

$$\varphi \colon \mathbb{R} \to \mathbb{C},\ a \in \mathbb{R} \mapsto \varphi(a) := \tilde{a},$$

is a *field isomorphism* of $\mathbb{R}$ onto its image in $\mathbb{C}$.

Accordingly, one “identifies” the image of $\mathbb{R}$ under φ with $\mathbb{R}$, writes a instead of $\tilde{a}$, and considers $\mathbb{R}$ as subset of $\mathbb{C}$.

Now let $i := (0, 1)$. Then for $a, b \in \mathbb{R}$ we have

$$\begin{aligned} (a,b) &= (a,0) + (0,b) \\ &= (a,0) + (b,0) \cdot (0,1) \\ &= \tilde{a} + \tilde{b}i \;=\; a + bi. \end{aligned}$$

So every complex number $z \in \mathbb{C}$ can uniquely be written as $z = a + bi$, where $a, b \in \mathbb{R}$. Note also that we have

$$\begin{aligned} i^2 := i \cdot i &= (0,1) \cdot (0,1) \\ &= (-1, 0) \\ &= \widetilde{-1} = -1. \end{aligned}$$

Using these conventions we reconcile the precise definition of complex numbers and their operations with the intuitive notion that we took as out starting point.

Definition 1.5. Let $z = a + bi \in \mathbb{C}$, where $a, b \in \mathbb{R}$. We define

(i) $\mathrm{Re}(z) := a$ (the *real part* of z),

(ii) $\mathrm{Im}(z) := b$ (the *imaginary part* of z),

(iii) $\bar{z} := a - bi$ (the *complex conjugate* of z),

(iv) $|z| := \sqrt{a^2 + b^2}$ (the *absolute value* of z).

1.6. Geometric interpretations. These concepts and also addition of complex numbers have obvious geometric interpretations if one identifies $z = a + bi$ with the point (a, b) in the plane $\mathbb{R}^2$.

1.7. Subtraction and division. As in every field one can define a notion of *subtraction* and *division* of complex numbers. Namely, if $z \in \mathbb{C}$, one denotes the additive inverse of z by $-z$ and defines

$$w - z := w + (-z)$$

for $z, w \in \mathbb{C}$. If $z = a + bi$, $w = c + di$, then $-z = (-a) + (-b)i$, and so

$$w - z = (c - a) + (d - b)i.$$

If $z \neq 0$, then we denote by z^{-1} the *(multiplicative) inverse* of z. For $z = a + bi \neq 0$ we have

$$z^{-1} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i.$$

One defines

$$z/w = \frac{z}{w} := w \cdot z^{-1}.$$

One can compute with fractions of complex numbers as usual (as in any field). Using the fact that $z\bar{z} = |z|^2$, one can simplify fractions of complex numbers by multiplying in numerator and denominator by the complex conjugate of the denominator. For example,

$$\begin{aligned} \frac{2+i}{3+5i} &= \frac{(2+i)(3-5i)}{(3+5i)(3-5i)} \\ &= \frac{(6+5) + (-10+3)i}{3^2 + 5^2} \\ &= \frac{11 - 7i}{34} = \frac{11}{34} - \frac{7}{34} i. \end{aligned}$$

Theorem 1.8. *Let $z, w \in \mathbb{C}$. Then*

(i) $\mathrm{Re}(z) = \frac{1}{2}(z + \bar{z})$,

(ii) $\mathrm{Im}(z) = \frac{1}{2i}(z - \bar{z})$,

(iii) $\mathrm{Re}(z + w) = \mathrm{Re}(z) + \mathrm{Re}(w)$,

(iv) $\mathrm{Im}(z + w) = \mathrm{Im}(z) + \mathrm{Im}(w)$,

(v) $z \in \mathbb{R}$ *iff* $\mathrm{Im}(z) = 0$ *iff* $z = \bar{z}$,

(vi) $\bar{\bar{z}} = z$,

(vii) $\overline{z + w} = \bar{z} + \bar{w}$, $\overline{z - w} = \bar{z} - \bar{w}$,

(viii) $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$,

(ix) $\overline{\left(\dfrac{z}{w}\right)} = \dfrac{\bar{z}}{\bar{w}}$,

(x) $z \cdot \bar{z} = |z|^2$,

(xi) $|z| = 0$ *iff* $z = 0$,

(xii) $|z \cdot w| = |z| \cdot |w|$,

(xiii) $\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$,

(xiv) $|z + w| \leq |z| + |w|$ (triangle inequality).

Proof. The proofs of these facts are straightforward, often tedious, and we omit the details; as an example, we will prove (xii).

If $z = a + bi$ and $w = c + di$, where $a, b, c, d \in \mathbb{R}$, then

$$z \cdot w = (ac - bd) + (ad + bc)i,$$

and so

$$\begin{aligned}|z \cdot w|^2 &= (ac - bd)^2 + (ad + bc)^2 \\ &= a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2 \\ &= (a^2 + b^2)(c^2 + d^2) \\ &= |z|^2 \cdot |w|^2.\end{aligned}$$

Hence

$$|z \cdot w| = |z| \cdot |w|.$$

(For this conclusion it is important that the terms on both sides are non-negative). $\square$

Definition 1.9 (The exponential function for complex arguments)**.** For $z = x + iy$, where $x, y \in \mathbb{R}$, we define

$$e^z = \exp(z) := e^x(\cos y + i \sin y).$$

Note that for $z \in \mathbb{R}$, when $y = \mathrm{Im}(z) = 0$, this agrees with the usual exponential function. So the exponential function for complex arguments is an extension of the exponential function for real arguments. Choosing this particular extension seems arbitrary and unmotivated at this point. It will become more natural after we have introduced power series, because we will see that the complex exponential function can be represented by the usual power series for the real exponential function. A deeper reason for choosing this particular extension is that the complex exponential function is *holomorphic* and by the *uniqueness theorem* every function on $\mathbb{R}$ has at most one holomorphic extension to $\mathbb{C}$. All this will be discussed later in the course.

Theorem 1.10. *The exponential function* $\exp\colon \mathbb{C} \to \mathbb{C}$ *has the following properties:*

(i) $e^{it} = \cos t + i \sin t$ *for* $t \in \mathbb{R}$ (Euler-Moivre formula)*,*

(ii) $e^{z+w} = e^z \cdot e^w$ *for* $z, w \in \mathbb{C}$ (functional equation of exp)*,*

(iii) $e^{z+2\pi i} = e^z$ *for* $z \in \mathbb{C}$ (exp is $2\pi i$-periodic)*,*

(iv) $e^z = 1$ *iff* $z = 2\pi ik$ *with* $k \in \mathbb{Z}$*,*

(v) $e^w = e^z$ *iff* $w = z + 2\pi i k$ *with* $k \in \mathbb{Z}$.

Proof. (i) Obvious from the definition.

(ii) If $z = x + iy$ and $w = u + iv$ with $x, y, u, v \in \mathbb{R}$, then

$$z + w = (x + u) + i(y + v).$$

Note that

$$\begin{aligned} \cos(y + v) &= \cos y \cos v - \sin y \sin v, \\ \sin(y + v) &= \sin y \cos v + \cos y \sin v. \end{aligned}$$

Hence

$$\begin{aligned} e^z \cdot e^w &= e^x(\cos y + i \sin y)e^u(\cos v + i \sin v), \\ &= e^{x+u}\big((\cos y \cos v - \sin y \sin v) + i(\sin y \cos v + \cos y \sin v)\big) \\ &= e^{x+u}(\cos(y + v) + i \sin(y + v)) \\ &= e^{(x+u)+i(y+v)} = e^{z+w}. \end{aligned}$$

(iv) Let $z = x + iy$, $x, y \in \mathbb{R}$. Then

$$\begin{aligned} e^z = 1 \quad &\Leftrightarrow \quad e^x(\cos y + i \sin y) = 1 \\ &\Leftrightarrow \quad e^x \cos y = 1 \text{ and } e^x \sin y = 0, \\ &\Leftrightarrow \quad e^x \cos y = 1 \text{ and } \sin y = 0, \\ &\Leftrightarrow \quad e^x \cos y = 1 \text{ and } y = n\pi \text{ for } n \in \mathbb{Z}, \\ &\Leftrightarrow \quad e^x(-1)^n = 1 \text{ and } y = n\pi \text{ for } n \in \mathbb{Z}, \\ &\Leftrightarrow \quad e^x = 1 \text{ and } y = n\pi \text{ for some } n \in \mathbb{Z} \text{ even}, \\ &\Leftrightarrow \quad x = 0 \text{ and } y = n\pi \text{ for some } n \in \mathbb{Z} \text{ even}, \\ &\Leftrightarrow \quad z = x + iy = 2\pi i k \text{ for } k \in \mathbb{Z}. \end{aligned}$$

(iii) Using (ii) and (iv) we have

$$e^{z+2\pi i} = e^z \cdot e^{2\pi i} = e^z \cdot 1 = e^z$$

for $z \in \mathbb{C}$.

(v) Note that $e^z \neq 0$ for $z \in \mathbb{C}$, because $e^z \cdot e^{-z} = e^0 = 1 \neq 0$. So for $z, w \in \mathbb{C}$ we have by (iv),

$$\begin{aligned} e^w = e^z \quad &\Leftrightarrow \quad e^{w-z} = e^w \cdot e^{-z} = e^z \cdot e^{-z} = e^0 = 1, \\ &\Leftrightarrow \quad w - z = 2\pi i k \text{ for some } k \in \mathbb{Z}, \\ &\Leftrightarrow \quad w = z + 2\pi i k \text{ for some } k \in \mathbb{Z}. \end{aligned}$$

□

1.11. Mapping properties of exp**.** The exponential function maps lines parallel to the real axis to rays starting at 0; lines parallel to the imaginary axis are mapped to circles centered at 0.

The exponential function maps the strip

$$S = \{x + iy : x \in \mathbb{R},\, 0 < y < 2\pi\}$$

bijectively onto $\mathbb{C} \setminus [0, \infty)$.

1.12. Polar coordinates. If $z = x + iy$, $x, y \in \mathbb{R}$, then by using polar coordinates we can write

$$\begin{aligned} x &= r \cos\varphi, \\ y &= r \sin\varphi, \end{aligned}$$

where $r \geq 0$ and $\varphi \in \mathbb{R}$. Hence every complex number can be written as

$$z = x + iy = r(\cos\varphi + i \sin\varphi) = re^{i\varphi},$$

where $r \geq 0$ and $\varphi \in \mathbb{R}$. Note that $r = |z|$ is the absolute value of z. The angle φ is called the *argument* of z, written $\varphi = \arg(z)$. It is only determined up to integer multiples of 2π. If $\mathrm{Re}(z) \neq 0$, then

$$\tan\varphi = \frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}.$$

This formular allows the computation of φ for given z.

1.13. Geometric interpretation of multiplication and division of complex numbers. Let $z = re^{i\alpha}$ and $w = se^{i\beta}$, where $r, s \geq 0$ and $\alpha, \beta \in \mathbb{R}$. Then

$$z \cdot w = re^{i\alpha} se^{i\beta} = rse^{i(\alpha+\beta)},$$

and

$$\frac{z}{w} = \frac{re^{i\alpha}}{se^{i\beta}} = \frac{r}{s}e^{i\alpha}e^{-i\beta} = \frac{r}{s}e^{i(\alpha-\beta)}, \quad w \neq 0.$$

So complex numbers are mutliplied by mutiplying their absolute values and adding their arguments. One divides them by dividing their absolute values and subtracting their arguments.

These facts and their geometric interpretations will be used throughout; for example, multiplication by $i = e^{i\pi/2}$, i.e., the map $z \in \mathbb{C} \mapsto iz$ correspond to counterclockwise rotation in the plane by 90^o.

1.14. Computation of nth roots. For $z \in \mathbb{C}$ and $n \in \mathbb{N}$ we set

$$z^n := \underbrace{z \cdots z}_{n \text{ factors}}.$$

We use the convention $z^0 := 1$ for $z \in \mathbb{C}$, and set $z^n := (z^{-1})^{|n|}$ for $n \in \mathbb{Z}$, $n < 0$, $z \neq 0$. One then has the usual computational rules

$$\begin{aligned} z^n z^k &= z^{n+k}, \\ (z^n)^k &= z^{nk}, \\ (zw)^n &= z^n w^n \end{aligned}$$

for $z, w \in \mathbb{C}$, $z, w \neq 0$, $n, k \in \mathbb{Z}$.

Now let $n \in \mathbb{N}$, $a \in \mathbb{C}$, $a \neq 0$. Every solution z of the equation $z^n = a$ (for given n and a) is called an *nth root* of a. As we will see momentarily, every $a \neq 0$ has precisely n distinct nth roots; hence for complex numbers a we will usually not use the ambiguous notation $\sqrt[n]{a}$, but we will use it for positive real numbers a (where it denotes the unique positive real number with $(\sqrt[n]{a})^n = a$).

We write $a = re^{i\varphi}$ with $r > 0$, $\varphi \in \mathbb{R}$, and use the *ansatz* $z = \rho e^{i\alpha}$, where $\rho > 0$ and $\alpha \in \mathbb{R}$. Then

$$z^n = \underbrace{z \cdots z}_{n \text{ factors}} = \rho^n e^{in\alpha} = re^{i\varphi}.$$

Hence $\rho^n = r$ and $n\alpha = \varphi + 2\pi k$ for $k \in \mathbb{Z}$. This implies $\rho = \sqrt[n]{r}$ and

$$\alpha = \frac{\varphi}{n} + \frac{2\pi}{n}k, \quad k \in \mathbb{Z}.$$

We conclude that a complex number $a = re^{i\varphi}$, $a \neq 0$, has n distinct nth roots

$$z_k = \sqrt[n]{r}e^{i\alpha_k},$$

where

$$\alpha_k = \frac{\varphi}{n} + \frac{2\pi}{n}k \quad \text{with} \quad k \in \{0, \dots, n-1\}.$$

1.15. Examples. (a) Third roots of $a = -8$: We have $a = -8 = 8e^{i\pi}$; so

$$z_k = \sqrt[3]{8}e^{i\alpha_k} = 2e^{i\alpha_k},$$

$$\alpha_k = \tfrac{\pi}{3} + \tfrac{2\pi}{3}k \quad \text{with} \quad k \in \{0, 1, 2\}.$$

Hence

$$\alpha_0 = \tfrac{\pi}{3}, \quad \alpha_1 = \pi, \quad \alpha_2 = \tfrac{5\pi}{3},$$

and

$$\begin{aligned} z_0 &= 2e^{i\pi/3} = 2(\cos\tfrac{\pi}{3} + i\sin\tfrac{\pi}{3}) = 2(\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}) = 1 + i\sqrt{3}, \\ z_1 &= 2e^{i\pi} = -2, \\ z_2 &= 2e^{i5\pi/3} = 2(\cos\tfrac{5\pi}{3} + i\sin\tfrac{5\pi}{3}) = 2(-\tfrac{1}{2} - i\tfrac{\sqrt{3}}{2}) = -1 - i\sqrt{3}. \end{aligned}$$

(b) Computation of square roots by a different method: To solve the equation

$$z^2 = -3 + 4i,$$

for example, we use the *ansatz* $z = a + bi$ with $a, b \in \mathbb{R}$ and solve for a and b:

$$z^2 = (a + bi)^2 = a^2 - b^2 + 2abi = -3 + 4i.$$

Hence

$$a^2 - b^2 = -3 \quad \text{and} \quad 2ab = 4. \tag{1}$$

Squaring both equations and adding leads to

$$(a^2 - b^2)^2 + 4a^2b^2 = a^4 + 2a^2b^2 + b^4 = (a^2 + b^2)^2 = (-3)^2 + 4^2 = 25.$$

Thus,

$$a^2 + b^2 = 5.$$

Combining this with (1) gives

$$2a^2 = 2 \quad \Rightarrow \quad a^2 = 1 \quad \Rightarrow \quad a_1 = 1,\ a_2 = -1$$

and
$$b_1 = 2,\ b_2 = -2.$$
So we get the solutions
$$z_1 = 1 + 2i \quad \text{and} \quad z_2 = -1 - 2i.$$
Note that $z_2 = -z_1$ as it should be.

(c) Solutions of quadratic equations can be computed as usual by completing the square, etc. A more general fact is true: Every polynomial equation
$$z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = 0,$$
where $a_0, \ldots, a_{n-1} \in \mathbb{C}$ has a solution $z \in \mathbb{C}$. This *Fundamental Theorem of Algebra* will be proved later in this course.

2 Topological properties of $\mathbb{C}$

In this section we summarize some standard facts from point-set topology. We will mostly omit proofs or only give a brief outline.

Definition 2.1. A *metric space* (X, d) is a set X together with a function $d\colon X \times X \to [0, \infty)$ such that for all $x, y, z \in X$ the following properties are true:

(i) $d(x, y) = 0$ iff $x = y$,

(ii) $d(x, y) = d(y, x)$ (symmetry),

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

If these axioms hold, then d is called a *metric* on X.

In a metric space (X, d) we define for $a \in X$ and $r > 0$,

$$B(a, r) := \{x \in X : d(x, a) < r\},$$
$$\overline{B}(a, r) := \{x \in X : d(x, a) \leq r\},$$

and call this the *open ball* and the *closed ball* of radius r centered at a, respectively.

Often one also calls $B(a, r)$ the *(open) r-neighborhood* of a.

Example 2.2. If we define $d(z, w) := |z - w|$ for $z, w \in \mathbb{C}$, then d is a metric on $\mathbb{C}$, called the *Euclidean metric*. From now on we consider $\mathbb{C}$ as a metric space equipped with the Euclidean metric.

Definition 2.3. Let (X, d) be a metric space, and $M \subseteq X$. A point $x \in X$ is called

(i) an *interior point* of M if there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq M$,

(ii) an *exterior point* of M if there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq X \setminus M$.

(iii) a *boundary point* of M if $M \cap B(x, \epsilon) \neq \emptyset$ and $(X \setminus M) \cap B(x, \epsilon) \neq \emptyset$ for all $\epsilon > 0$.

The set of interior points of M is denoted by $\mathrm{int}(M)$, and the set of boundary points by ∂M.

Remark 2.4. Every point in X is an interior point, an exterior point, or a boundary point of M, and these cases are mutually exclusive. An interior point of M always belongs to M, while an exterior point always lies in the complement of M in X. A boundary point of M may or may not belong to M.

Definition 2.5. Let (X, d) be a metric space, and $M \subseteq X$. Then M is called *open* if it has only interior points, i.e., for all $x \in M$ there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq M$.

The set M is called *closed* if it contains all of its boundary points.

Example 2.6. $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ is an open set in $\mathbb{C}$, called the *open unit disk*. The set $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$ is a closed set in $\mathbb{C}$, called the *closed unit disk*. The set $M = \mathbb{D} \cup \{1\}$ is neither open nor closed.

Definition 2.7. Let (X, d) be a metric space, and $M \subseteq X$. Then $\overline{M} := M \cup \partial M$ is called *the closure* of M.

One can show that $\overline{M}$ is the smallest closed set in X containing M.

Our notation for the closed unit disk $\overline{\mathbb{D}}$ was motivated by the fact that this set is the closure of $\mathbb{D}$.

Theorem 2.8. *Let (X, d) be a metric space. Then the following statements are true:*

(i) *a set in X is* $\left\{\begin{array}{c}\textit{open}\\ \textit{closed}\end{array}\right\}$ *if its complement is* $\left\{\begin{array}{c}\textit{closed}\\ \textit{open}\end{array}\right\}$,

(ii) $\left\{\begin{array}{c}\textit{a union}\\ \textit{an intersection}\end{array}\right\}$ *of a family of* $\left\{\begin{array}{c}\textit{open}\\ \textit{closed}\end{array}\right\}$ *sets in X is* $\left\{\begin{array}{c}\textit{open}\\ \textit{closed}\end{array}\right\}$,

(iii) $\left\{\begin{array}{c}\textit{an intersection}\\ \textit{a union}\end{array}\right\}$ *of a finite family of* $\left\{\begin{array}{c}\textit{open}\\ \textit{closed}\end{array}\right\}$ *sets in X is* $\left\{\begin{array}{c}\textit{open}\\ \textit{closed}\end{array}\right\}$.

Remark 2.9. Suppose X is a set together with a family $\mathcal{O}$ of its subsets called *open sets*. If $\emptyset, X \in \mathcal{O}$ and if the properties (ii) and (iii) in the previous theorem are satisfied, then $(X, \mathcal{O})$ is called a *topological space* and the system $\mathcal{O}$ a *topology* on X.

By what we have seen, every metric d on a set X determines a natural system $\mathcal{O}$ of open sets that form a topology on X. One calls this the topology *induced* by d.

Definition 2.10. Let (X, d) be a metric space, and $\{x_n\}$ be a sequence of points in X.

(i) The sequence $\{x_n\}$ is called *convergent* if there exists a point $x \in X$ (the *limit* of $\{x_n\}$) such that for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$ we have $d(x, x_n) < \epsilon$.

One can show that if the limit x exists, then it is unique, and one writes $x = \lim_{n\to\infty} x_n$ or simply $x_n \to x$.

(ii) The sequence $\{x_n\}$ is called a *Cauchy sequence* if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n, k \in \mathbb{N}$ with $n, k \geq N$ we have $d(x_n, x_k) < \epsilon$.

(iii) A point $x \in X$ is called a *sublimit* of $\{x_n\}$ if there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = x$. In this case, we say that $\{x_n\}$ *subconverges* to x.

Proposition 2.11. *A sequence $\{z_n\}$ in $\mathbb{C}$ converges if and only if the sequences $\{\operatorname{Re}(z_n)\}$ and $\{\operatorname{Im}(z_n)\}$ of real numbers converge.*

In case of convergence we have

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} \operatorname{Re}(z_n) + i \lim_{n\to\infty} \operatorname{Im}(z_n).$$

Proof. $\Rightarrow$: Suppose $\{z_n\}$ converges, and let $z := \lim_{n\to\infty} z_n$. Note that $|\operatorname{Re}(w)| \leq |w|$ for $w \in \mathbb{C}$. Hence

$$|\operatorname{Re}(z_n) - \operatorname{Re}(z)| \leq |z_n - z|$$

for $n \in \mathbb{N}$. This implies that $\lim_{n\to\infty} \operatorname{Re}(z_n) = \operatorname{Re}(z)$. Similarly, $\lim_{n\to\infty} \operatorname{Im}(z_n) = \operatorname{Im}(z)$.

$\Leftarrow$: Suppose $\{\operatorname{Re}(z_n)\}$ and $\{\operatorname{Im}(z_n)\}$ converge. Let $a := \lim_{n\to\infty} \operatorname{Re}(z_n)$, $b := \lim_{n\to\infty} \operatorname{Im}(z_n)$, and $z := a + bi$. Note that for all $w \in \mathbb{C}$ we have

$$|w| = \sqrt{\operatorname{Re}(w)^2 + \operatorname{Im}(w)^2} \leq |\operatorname{Re}(w)| + |\operatorname{Im}(w)|.$$

Hence

$$|z_n - z| \leq |\operatorname{Re}(z_n) - a| + |\operatorname{Im}(z_n) - b|.$$

It follows that $\lim_{n\to\infty} z_n = z$. $\square$

Remark 2.12. The usual computational rules for limits are true for sequences in $\mathbb{C}$. If $\{z_n\}$ and $\{w_n\}$ are sequences in $\mathbb{C}$ with $z_n \to z$ and $w_n \to w$, then

(i) $z_n + w_n \to z + w$,

(ii) $z_n w_n \to zw$,

(iii) $\mathrm{Re}(z_n) \to \mathrm{Re}(z)$, $\mathrm{Im}(z_n) \to \mathrm{Im}(z)$, $\overline{z}_n \to \overline{z}$, $|z_n| \to |z|$,

(iv) $\dfrac{w_n}{z_n} \to \dfrac{w}{z}$, if, in addition, $z \neq 0$ and $z_n \neq 0$ for $n \in \mathbb{N}$.

Definition 2.13. A metric space (X, d) is called *complete* if every Cauchy sequence in X converges.

Theorem 2.14. *The space $\mathbb{C}$ (equipped with the Euclidean metric) is complete.*

Proof. This follows from the completeness of $\mathbb{R}$ and Proposition 2.11. □

Proposition 2.15. *A subset A of a metric space (X, d) is closed if and only if every convergent sequence in A has its limit also in A.*

Definition 2.16. A subset K of a metric space (X, d) is called *compact* if every open cover of K has a finite subcover; i.e., whenever $\{U_i : i \in I\}$ is a family of open sets in X with $K \subseteq \bigcup_{i\in I} U_i$, then there exist $i_1, \dots, i_n \in I$ such that $K \subseteq U_{i_1} \cup \dots \cup U_{i_n}$.

Theorem 2.17. *Let (X, d) be a metric space. Then $K \subseteq X$ is compact if and only if every sequence $\{x_n\}$ in K has a sublimit in K, i.e., there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges and $x = \lim\limits_{n\to\infty} x_n \in K$.*

The last condition is called *sequential compactness*. So a set in a metric space is compact if and only if it is sequentially compact.

Proposition 2.18. (a) *Every compact metric space is complete.*

(b) *Every compact subset of a metric space is closed.*

Theorem 2.19 (Heine-Borel)**.** *A subset $K \subseteq \mathbb{C}$ is compact if and only if it is closed and bounded.*

Remark 2.20. A subset M of a metric space is called *bounded* if its *diameter* defined as

$$\operatorname{diam}(M) := \sup\{d(x,y) : x, y \in M\}$$

is finite. This is equivalent to the requirement that there exists $a \in X$ and $r > 0$ such that $M \subseteq B(a,r)$.

Definition 2.21. Let (X,d) and (Y,ρ) be metric spaces, and $f\colon X \to Y$ be a map.

(i) We say that f *approaches the limit* $y \in Y$ *as* x *approaches* $a \in X$, written as $\lim_{x\to a} f(x) = y$, if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in X$ we have that

$$0 < d(x,a) < \delta \quad \text{implies} \quad \rho(f(x), y) < \epsilon.$$

(ii) The map f is called *continuous at* $a \in X$ if $\lim_{x\to a} f(x) = f(a)$.

(iii) The map f is called *continuous (on X)* if it is continuous at all points $a \in X$.

Note that in (i) it does not matter what happens for $x = a$.

Proposition 2.22. *Let (X,d) and (Y,ρ) be metric spaces, and $f\colon X \to Y$ be a map. Then the following condition are equivalent:*

(i) *f is continuous,*

(ii) *for every convergent sequence $\{x_n\}$ in X, the sequence $\{f(x_n)\}$ also converges and we have*

$$\lim_{n\to\infty} f(x_n) = f\bigl(\lim_{n\to\infty} x_n\bigr),$$

(iii) *for all $x \in X$ and all $\epsilon > 0$ there exists $\delta > 0$ such that*

$$f(B(x,\delta)) \subseteq B(f(x), \epsilon),$$

(iv) *preimages of open sets are open, i.e.,* $f^{-1}(V)$ *is open in* X *whenever* $V \subseteq Y$ *is open in* Y*.*

(v) *preimages of closed sets are closed.*

Remark 2.23. We will be mostly interested in limits of functions $f\colon M \to \mathbb{C}$, where $M \subseteq \mathbb{R}$ or $M \subseteq \mathbb{C}$. In this case, one has the usual computational rules for function limits. For example, if $f\colon M \to \mathbb{C}$ and $g\colon M \to \mathbb{C}$ are functions, and $a \in M$, then

$$\lim_{x\to a} f(x)g(x) = \big(\lim_{x\to a} f(x)\big)\big(\lim_{x\to a} g(x)\big)$$

if the limits on the right hand side exist, etc. Based on this one can prove that if $f\colon M \to \mathbb{C}$ and $g\colon M \to \mathbb{C}$ are continuous, then $f+g$ is continuous, fg is continuous, etc.

Theorem 2.24. *Let* (X,d) *and* (Y,ρ) *be metric spaces, and* $f\colon X \to Y$ *be a continuous map. If* $K \subseteq X$ *is compact, then* $f(K) \subseteq Y$ *is compact (continuous images of compact sets are compact).*

Proof. Suppose that $K \subseteq X$ is compact, and let $\{y_n\}$ be an arbitrary sequence in $f(K)$. Then for all $n \in \mathbb{N}$ there exists $x_n \in K$ such that $y_n = f(x_n)$. Then $\{x_n\}$ is a sequence in K. Since K is compact, by Theorem 2.17 the sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ with a limit in K, say $x_{n_k} \to x \in K$. By continuity of f it follows that

$$y_{n_k} = f(x_{n_k}) \to f(x) \in f(K).$$

So $\{y_n\}$ has a sublimit in $f(K)$. Hence $f(K)$ is compact by Theorem 2.17. $\square$

Remark 2.25. There are other important statements involving compactness and continuity: a real-valued function $f\colon X \to \mathbb{R}$ on a compact metric space (X,d) attains maximum and minimum. A continuous map $f\colon X \to Y$ between a compact metric space (X,d) and a metric space (Y,ρ) is *uniformly continuous*, i.e., for all $\epsilon > 0$ that exists $\delta > 0$ such that for all $x, y \in X$ we have that

$$d(x,y) < \delta \quad \text{implies} \quad \rho(f(x), f(y)) < \epsilon.$$

Definition 2.26. A subset M of a metric space (X,d) is called *connected* if the following condition is true: if $U, V \subseteq X$ are open sets with $M \subseteq U \cup V$ and $U \cap V = \emptyset$, then $M \subseteq U$ or $M \subseteq V$.

Remark 2.27. (a) For $M = X$ this statement can be reformulated in equivalent form in the following way: a metric space X is connected if every decomposition $X = U \cup V$, $U \cap V = \emptyset$, into open subsets $U, V \subseteq X$ is trivial, i.e., $U = M$ or $V = M$. Equivalently, X is connected if $\emptyset$ and X are the only subsets of X that are both open and closed.

(b) For subsets M of a metric space (X, d) one can characterize connectedness in a similar way if one uses relatively closed and open sets. By definition a set $A \subseteq M$ is called *relatively closed* or *relatively open* in M if there exists a closed or an open set $B \subseteq X$, respectively, such that $A = B \cap M$.

The *restriction* $d_M := d|M \times M$ *of* d *to* M is a metric on M. It is not hard to see that $A \subseteq M$ is relatively closed or relatively open in M if A is closed or open in the metric space (M, d_M), respectively.

Using this terminology one can show that a subset M of a metric space is connected if and only if $\emptyset$ and M are the only subsets of M that are both relatively closed and relatively open in M (exercise!).

Proposition 2.28. *A non-empty subset $M \subseteq \mathbb{R}$ is connected if and only if M is an interval (possibly degenerate).*

Here we call an interval *degenerate* if it consists of only one point. All intervals in $\mathbb{R}$ are of the form $[a, b]$, $[a, b)$, $(a, b]$, (a, b), $(-\infty, a]$, $(-\infty, a)$, $[a, +\infty)$, $(a, +\infty)$, or $(-\infty, +\infty) = \mathbb{R}$, where $a, b \in \mathbb{R}$, $a \leq b$.

Proposition 2.29. *Let (X, d) and (Y, ρ) be metric spaces, and $f\colon X \to Y$ be a continuous map. If $M \subseteq X$ is connected, then $f(M) \subseteq Y$ is connected (continuous images of connected sets are connected).*

Proof. Under the given assumptions, assume that $M \subseteq X$ is connected. To show that $f(M)$ is connected, let $U, V \subseteq Y$ be arbitrary open sets with $f(M) \subseteq U \cup V$ and $U \cap V = \emptyset$. Since f is continuous, the sets $U' := f^{-1}(U)$ and $V' := f^{-1}(V)$ are open. Moreover,

$$U' \cup V' = f^{-1}(U) \cup f^{-1}(V) = f^{-1}(U \cup V) \supseteq f^{-1}(f(M)) \supseteq M,$$

and

$$U' \cap V' = f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\emptyset) = \emptyset.$$

Since M is connected, we have $M \subseteq U'$ or $M \subseteq V'$, say $M \subseteq U'$. Then

$$f(M) \subseteq f(U') = f(f^{-1}(U)) \subseteq U.$$

Hence $f(M)$ is connected. □

Definition 2.30. A non-empty set $\Omega \subseteq \mathbb{C}$ is called a *region* if it is open and connected.

Theorem 2.31. *A non-empty open set $\Omega \subseteq \mathbb{C}$ is a region if and only if any two points $a, b \in \Omega$ can be joined by a polygonal path in Ω.*

If $z, w \in \mathbb{C}$ we denote by

$$[z, w] := \{(1-t)z + tw : t \in [0,1]\}$$

the closed *line segment* with endpoints z and w. A *polygonal path* P in $\mathbb{C}$ is a set of the form

$$P = [z_0, z_1] \cup \cdots \cup [z_{n-1}, z_n],$$

where $n \in \mathbb{N}$, $z_1, \ldots, z_n \in \mathbb{C}$. Such a path *joins* two points z and w if $z_0 = z$ and $w = z_n$.

Proof. $\Rightarrow$: Assume that Ω is a region. Fix a point $a \in \Omega$, and consider the points set M of points $b \in \Omega$ that can be joined to a by a polygonal path in Ω. It suffices to show that $M = \Omega$.

Note that $a \in M$, and so $M \neq \emptyset$. In order to conclude that $M = \Omega$, it is enough to show that M is open and relatively closed in Ω.

1. M is open.

If $y \in M \subseteq \Omega$, then, since Ω is open, there exists $\epsilon > 0$ such that $B(y, \epsilon) \subseteq \Omega$. If $z \in B(y, \epsilon)$ we can find a polygonal path in Ω joining a and z as a union of a polygonal path joining a and y and the line segment $[y, z] \subseteq B(y, \epsilon) \subseteq \Omega$. Hence $B(y, \epsilon) \subseteq M$, and so M is open.

2. M is relatively closed.

Let $\{x_n\}$ be an arbitrary sequence in M that converges in the "ambient" space Ω. So $x_n \to x \in \Omega$. We have to show that $x \in M$.

Since Ω is open, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq \Omega$. Since $x_n \to x$, there exists $k \in \mathbb{N}$ such that $x_k \in B(x, \epsilon)$. Now $x_k \in M$ and $[x_k, x] \subseteq B(x, \epsilon)$. So we can find a polygonal path in Ω joining a and x by first joining a and x_k by a polygonal path in Ω followed by the line segment $[x_k, x]$. So $x \in M$, and M is relatively closed.

$\Leftarrow$: Suppose any two points in Ω can be joined by a polygonal path in Ω. Fix $a \in \Omega$. Then for every point $b \in \Omega$ there exists a polygonal path $P_b \subseteq \Omega$ joining a and b. Then $\Omega = \bigcup_{b \in \Omega} P_b$ and $a \in \bigcap_{b \in \Omega} P_b$. Moreover, each polygonal path is the image of an interval in $\mathbb{R}$ and hence connected.

We conclude that Ω is connected, because a union of a family of connected sets with non-empty intersection is connected (exercise!). $\square$

3 Differentiation

Remark 3.1. An obvious extension of differential calculus of real-valued functions of one real variable is to complex-valued functions of one real variable; so if $f\colon I \to \mathbb{C}$ is a complex-valued function on an interval $I \subseteq \mathbb{R}$, we call f *differentiable* at $x_0 \in I$, if the limit

$$\lim_{x \in x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. This limit is denoted by $f'(x_0)$ and called the *derivative of f at x_0*.

It (essentially) follows from Proposition 2.11 that f is differentiable at x_0 if and only if the functions $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are differentiable at x_0. In this case

$$f'(x_0) = (\mathrm{Re}(f))'(x_0) + i(\mathrm{Im}(f))'(x_0).$$

For example, let $f\colon \mathbb{R} \to \mathbb{C}$, $f(x) := e^{ix}$ for $x \in \mathbb{R}$. Then $f(x) = \cos x + i \sin x$, and so

$$\begin{aligned} f'(x) &= -\sin x + i \cos x \\ &= i(\cos x + i \sin x) \;=\; ie^{ix}. \end{aligned}$$

One is tempted to use the chain rule here:

$$\frac{d}{dx} e^{ix} = \left.\frac{de^z}{dz}\right|_{z=ix} \cdot \frac{d(ix)}{dx} = e^z|_{z=ix} \cdot i = ie^{ix}.$$

At the moment this is not justified, because we have not yet defined derivatives of complex-valued functions of a complex variable.

Definition 3.2. Let $U \subseteq \mathbb{C}$ be open, and $f\colon U \to \mathbb{C}$ be a function.

(i) f is called *differentiable at $z_0 \in U$* if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. This limit is called the *derivative of f at z_0* and denoted by $f'(z_0)$, $\dfrac{df}{dz}(z_0)$, etc.

(ii) f is called *holomorphic* (on U) if it is differentiable at every point $z_0 \in U$.

(iii) We denote by $H(U)$ the set of all holomorphic functions on U.

Theorem 3.3. *Let $U, V \subseteq \mathbb{C}$ be open sets.*

(a) *If $f \in H(U)$, then f is continuous on U.*

(b) *Let $a, b \in \mathbb{C}$, and $f, g \in H(U)$. Then*

  (i) $af + bg \in H(U)$, *and* $(af + bg)' = af' + bg'$,

  (ii) $fg \in H(U)$ *and* $(fg)' = f'g + fg'$ (product rule),

  (iii) $\dfrac{f}{g} \in H(U)$ *and* $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$ *if we assume in addition that* $g(z) \neq 0$ *for* $z \in U$ (quotient rule).

(c) *Let $f \in H(U)$, $g \in H(V)$, and $f(U) \subseteq V$. Then $g \circ f \in H(U)$ and* $(g \circ f)' = (g' \circ f) \cdot f'$ (chain rule).

Proof. The standard proofs from real analysis transfer to this setting. For illustration we will prove (a) and (c).

(a) Let $z_0 \in U$ be arbitary. Then

$$\begin{aligned}
\lim_{z \to z_0} (f(z) - f(z_0)) &= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}(z - z_0) \\
&= \left(\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}\right) \cdot \left(\lim_{z \to z_0} (z - z_0)\right) \\
&= f'(z_0) \cdot 0 = 0.
\end{aligned}$$

Hence

$$\lim_{z \to z_0} f(z) = f(z_0),$$

and so f is continuous at z_0.

(c) A function $h\colon W \to \mathbb{C}$ on an open set $W \subseteq \mathbb{C}$ is differentiable at $z_0 \in W$ if and only if there exists a constant $c \in \mathbb{C}$, and a function $s\colon W \to \mathbb{C}$ with $\lim_{z \to z_0} s(z) = 0$ such that

$$h(z) - h(z_0) = (z - z_0)(c + s(z))$$

for all $z \in W$. If this is true, then $c = h'(z_0)$.

Now suppose f and g are as in the statement, and let $z_0 \in U$ be arbitrary. Set $w_0 = f(z_0)$. Then there exist functions $s\colon U \to \mathbb{C}$ and $t\colon V \to \mathbb{C}$ with $\lim\limits_{z\to z_0} s(z) = 0$ and $\lim\limits_{w\to w_0} t(w) = 0$ such that

$$\begin{aligned} f(z) - f(z_0) &= (z - z_0)(f'(z_0) + s(z)), \quad z \in U,\\ g(w) - g(w_0) &= (w - w_0)(g'(w_0) + t(w)), \quad w \in V. \end{aligned}$$

Hence for all $z \in U$ we have

$$\begin{aligned} g(\underbrace{f(z)}_{w}) - g(\underbrace{f(z_0)}_{w_0}) &= (f(z) - f(z_0))\big(g'(w_0) + t(f(z))\big),\\ &= (z - z_0)\big(f'(z_0) + s(z)\big)\big(g'(w_0) + t(f(z))\big)\\ = (z - z_0)\big(g'(w_0)f'(z_0) \;&+\; \underbrace{g'(w_0)s(z) + f'(z_0)t(f(z)) + s(z)t(f(z))}_{r(z)}\big). \end{aligned}$$

Note that $f(z) \to f(z_0) = w_0$ as $z \to z_0$ by continuity of f, and so

$$\begin{aligned} \lim_{z\to z_0} r(z) &= \lim_{z\to z_0}\big(g'(w_0)s(z) + f'(z_0)t(f(z)) + s(z)t(f(z))\big)\\ &= g'(w_0)\lim_{z\to z_0} s(z) + f'(z_0)\lim_{w\to w_0} t(w) + \big(\lim_{z\to z_0} s(z)\big)\big(\lim_{w\to w_0} t(w)\big)\\ &= g'(w_0)\cdot 0 + f'(z_0)\cdot 0 + 0\cdot 0 \;=\; 0. \end{aligned}$$

Hence $g \circ f$ is differentiable at z_0 and

$$(g \circ f)'(z_0) = g'(w_0)f'(z_0) = g'(f(z_0))f'(z_0).$$

The claim follows. $\square$

Remark 3.4. Similar statements hold for complex-valued functions on intervals. One can also prove a chain rule for a function given by a (post-)composition of a complex-valued function on interval by a holomorphic function.

Theorem 3.5. *Let $\Omega \subseteq \mathbb{C}$ be a region, and $f \in H(\Omega)$. If $f' = 0$, then f is a constant function.*

Proof. Fix $z_0 \in \Omega$. We will show that $f(z) = f(z_0)$ for all $z \in \Omega$.

Let $z \in \Omega$ be arbitrary. By Theorem 2.31 there exists a polygonal path

$$P = [z_0, z_1] \cup \cdots \cup [z_{n-1}, z_n] \subseteq \Omega,$$

where $z_n = z$. So it is enough to show that f is constant on each segment $[u, v] \subseteq \Omega$.

To see this, let $u, v \in \Omega$ be arbitrary, and consider $h\colon [0,1] \to \mathbb{C}$, $h(t) := f(u + (v-u)t)$ for $t \in [0,1]$. Then by the chain rule,

$$h'(t) = f'(u + (v-u)t)(v-u) = 0$$

for all $t \in [0,1]$. Hence h is constant on $[0,1]$, and so f is constant on $[u, v]$. $\square$

Remark 3.6. By using the previous statements, one can produce a certain, although limited, supply of holomorphic functions; namely, it follows from the definitions that a constant function $z \mapsto c \in \mathbb{C}$, and the function $z \mapsto z$ are holomophic in $\mathbb{C}$, and we have

$$\frac{dc}{dz} = 0 \quad \text{and} \quad \frac{dz}{dz} = 1.$$

By induction it follows from the product rule that $z \mapsto z^n$ is holomorphic in $\mathbb{C}$ for all $n \in \mathbb{N}$, and

$$\frac{dz^n}{dz} = nz^{n-1}.$$

This implies that each polynomial P, i.e., each function $P\colon \mathbb{C} \to \mathbb{C}$ of the form

$$z \mapsto P(z) := a_0 + a_1 z + \cdots + a_n z^n,$$

where $n \in \mathbb{N}_0$, $a_0, \ldots, a_n \in \mathbb{C}$, is holomorphic on $\mathbb{C}$. Finally, every rational function

$$R(z) = \frac{P(z)}{Q(z)},$$

where P and $Q \neq 0$ are polynomials, is holmorphic on the complement $\mathbb{C} \setminus Q^{-1}(0)$ of the zero-set of Q.

Remark 3.7. For the rest of this section we identify $\mathbb{C}$ with $\mathbb{R}^2$ by the correspondence $z = x + iy \in \mathbb{C} \cong (x, y) \in \mathbb{R}^2$. We write a complex-valued function $f\colon U \to \mathbb{C}$ on $U \subseteq \mathbb{C}$ in the form $f = u + iv$, where $u = \mathrm{Re}(f)$ and

$v = \mathrm{Im}(f)$. Then u and v are considered as functions of $(x, y) \in U \subseteq \mathbb{R}^2$, and we have

$$f(x + iy) = u(x, y) + iv(x, y),$$

where $x + iy \cong (x, y) \in U$.

We will denote by $u_x, u_y, \dots$ partial derivatives; so $u_x = \dfrac{\partial u}{\partial x}$, etc.

Theorem 3.8. *Let $U \subseteq \mathbb{C}$ be open, $f : U \to \mathbb{C}$, $f = u + iv$.*

Then f is holomorphic on U if and only if u and v are C^1-smooth on U and the so-called Cauchy-Riemann equations

$$u_x = v_y, \quad u_y = -v_x$$

are valid on U.

In this case,

$$\begin{aligned} f'(x + iy) &= u_x(x, y) + iv_x(x, y) \\ &= \frac{1}{i}(u_y(x, y) + iv_y(x, y)), \end{aligned}$$

whenever $x + iy \in U$.

For the proof of the implication $\Rightarrow$ we will use the following

Fact: *If $U \subseteq \mathbb{C}$ is open, and $f \in H(U)$, then $f' \in H(U)$.*

This will be proved later in this course (independently of the previous theorem of course).

Proof. $\Rightarrow$: Suppose f is holomorphic on U. Then for each $z = x + iy \in U$ we have,

$$f'(z) = \lim_{w \to 0} \frac{f(z + w) - f(z)}{w}.$$

Setting $w = h$ or $w = ik$, where $h, k \in \mathbb{R}$, and $h \to 0$ or $k \to 0$, we obtain

$$\begin{aligned} f'(x + iy) &= \lim_{h \to 0} \frac{f(z + h) - f(z)}{h} \\ &= \lim_{h \to 0} \frac{u(x + h, y) - u(x, y)}{h} + i \lim_{h \to 0} \frac{v(x + h, y) - v(x, y)}{h} \\ &= u_x(x, y) + iv_x(x, y), \end{aligned}$$

and

$$\begin{aligned} f'(x+iy) &= \lim_{k\to 0}\frac{f(z+ik)-f(z)}{ik} \\ &= \frac{1}{i}\lim_{k\to 0}\frac{u(x,y+k)-u(x,y)}{k}+\frac{1}{i}\cdot i\lim_{k\to 0}\frac{v(x,y+k)-v(x,y)}{k} \\ &= \frac{1}{i}(u_y(x,y)+iv_y(x,y)). \end{aligned}$$

In particular, the partial derivatives of u and v exist, and

$$u_x+iv_x=\frac{1}{i}(u_y+iv_y)=v_y-iu_y$$

on U; so $u_x=v_y$ and $v_x=-u_y$.

Moreover, we have that

$$u_x=\operatorname{Re}(f'),\quad v_x=\operatorname{Im}(f'),\quad u_y=-\operatorname{Im}(f'),\quad v_y=\operatorname{Re}(f').$$

Since by the fact mentioned above, f' is holomorphic and hence continuous on U, the partial derivatives of u and v are also continuous on U. It follows that u and v are C^1-smooth.

$\Leftarrow$: Suppose that u and v are C^1-smooth and that the Cauchy-Riemann equations hold. Since the functions u and v are C^1-smooth, they are differentiable on U; this means that for each point $(x,y)\in U$ and $(h,k)\in\mathbb{R}^2$ with $|(h,k)|:=\sqrt{h^2+k^2}$ small we have

$$u(x+h,y+k)=u(x,y)+u_x(x,y)h+u_y(x,y)k+r(h,k),$$

$$v(x+h,y+k)=v(x,y)+v_x(x,y)h+v_y(x,y)k+s(h,k),$$

where

$$\frac{r(h,k)}{|(h,k)|}\to 0\quad\text{and}\quad\frac{s(h,k)}{|(h,k)|}\to 0$$

as $|(h,k)|\to 0$.

If we set $z=x+iy$, $w=h+ik$, and $c=u_x(x,y)+iv_x(x,y)$, then for small $|w| \neq 0$, we have by the Cauchy-Riemann equations,

$$\left|\frac{f(z+w)-f(z)}{w} - c\right| = \left|\frac{f(z+w)-f(z)-(h+ik)c}{w}\right|$$

$$= \frac{1}{|w|}\big|f(z+w)-f(z)-hu_x(x,y)-ihv_x(x,y)-ku_y(x,y)-ikv_y(x,y)\big|$$

$$\leq \frac{1}{|w|}\big|u(x+h,y+k)-u(x,y)-hu_x(x,y)-ku_y(x,y)\big| +$$

$$\frac{1}{|w|}\big|v(x+h,y+k)-v(x,y)-hv_x(x,y)-kv_y(x,y)\big|$$

$$\leq \frac{|r(h,k)|+|s(h,k)|}{|(h,k)|} \to 0 \quad \text{as} \quad w = h+ik \to 0.$$

It follows that f is differentiable at z. Since $z \in U$ was arbitrary, f is holomorphic on U. $\square$

Example 3.9. By using the previous theorem, one can see that the exponential function is holomorphic on X; indeed,

$$e^z = e^{x+iy} = e^x \cos y + ie^x \sin y,$$

and so

$$u(x,y) = e^x \cos y \quad \text{and} \quad v(x,y) = e^x \sin y.$$

Hence u and v are C^1-smooth in $\mathbb{R}^2$. Moreover, we have

$$u_x = e^x \cos y = v_y \quad \text{and} \quad u_y = -e^x \sin y = -v_x,$$

and so the Cauchy-Riemann equations hold.

For the derivative of the exponential function we obtain

$$\frac{d}{dz}e^z = u_x + iv_x = e^x \cos y + ie^x \sin y = e^z.$$

Remark 3.10. If $f = u + iv$ is holomorphic, then u and v are C^1-smooth and the first partial derivatives of u and v are again real and imaginary parts of a holomorphic function ($u_x = \mathrm{Re}(f')$, etc.). It follows that the first partial derivatives of u and v are C^1-smooth, and so u and v are C^2-smooth. If we repeat this argument, we conclude that u and v are C^∞-smooth.

Definition 3.11. Let $U \subseteq \mathbb{R}^2$ be open, and $h\colon U \to \mathbb{R}$ be C^2-smooth. The *Laplacian* Δh of h is defined as

$$\Delta h = h_{xx} + h_{yy} = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}.$$

The function h is called *harmonic* on U if its Laplacian vanishes on U, i.e., if $\Delta h = 0$ on U.

Theorem 3.12. *Let $U \subseteq \mathbb{C}$ be open, $f \in H(U)$, $f = u + iv$. Then u and v are harmonic on U.*

Proof. The functions u and v are C^∞-smooth by Remark 3.10. Moreover, $u_x = v_y$ and $u_y = -v_x$ by the Cauchy-Riemann equations. Hence

$$\begin{aligned}\Delta u &= u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0,\\ \Delta v &= v_{xx} + v_{yy} = -u_{yx} + u_{xy} = 0.\end{aligned}$$

□

Remark 3.13. If $f = u+iv$ is a complex-valued function of $(x, y) \in U \subseteq \mathbb{R}^2$, one defines its partial derivatives as

$$\frac{\partial f}{\partial x} := \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \quad \text{and} \quad \frac{\partial f}{\partial y} := \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}.$$

Usually, it is better to work with the so-called *z- and $\bar{z}$-derivatives* or *Wirtinger derivatives* defined as

$$\frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \quad \text{and} \quad \frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

So symbolically,

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Note that

$$\begin{aligned}\frac{\partial z}{\partial z} &= \frac{1}{2}\left(\frac{\partial(x+iy)}{\partial x} - i\frac{\partial(x+iy)}{\partial y}\right) = \frac{1}{2}(1+1) = 1,\\ \frac{\partial \bar{z}}{\partial z} &= \frac{1}{2}\left(\frac{\partial(x-iy)}{\partial x} - i\frac{\partial(x-iy)}{\partial y}\right) = \frac{1}{2}(1-1) = 0.\end{aligned}$$

Similarly,

$$\frac{\partial \bar{z}}{\partial z} = 0 \quad \text{and} \quad \frac{\partial \bar{z}}{\partial \bar{z}} = 1.$$

So with respect to the operator $\dfrac{\partial}{\partial z}$ the function $z \mapsto z$ behaves as expected, and $z \mapsto \bar{z}$ like a constant, and we have a similar behavior for the operator $\dfrac{\partial}{\partial \bar{z}}$.

Moreover, the usual computational rules hold for these operators (product rule, quotient rule, etc.). This allows computations such as

$$\frac{\partial |z|^2}{\partial z} = \frac{\partial (z\bar{z})}{\partial z} = \frac{\partial z}{\partial z}\bar{z} + \frac{\partial \bar{z}}{\partial z} z = \bar{z}.$$

We also have a version of the chain rule: suppose that $h = h(w, \bar{w})$ and $w = w(z, \bar{z})$. Then

$$\frac{\partial h}{\partial z} = \frac{\partial h}{\partial w} \cdot \frac{\partial w}{\partial z} + \frac{\partial h}{\partial \bar{w}} \cdot \frac{\partial \bar{w}}{\partial z}, \quad \text{and} \quad \frac{\partial h}{\partial \bar{z}} = \frac{\partial h}{\partial w} \cdot \frac{\partial w}{\partial \bar{z}} + \frac{\partial h}{\partial \bar{w}} \cdot \frac{\partial \bar{w}}{\partial \bar{z}}.$$

Note also

$$\overline{\left(\frac{\partial w}{\partial z}\right)} = \frac{\partial \bar{w}}{\partial \bar{z}}, \quad \text{and} \quad \overline{\left(\frac{\partial w}{\partial \bar{z}}\right)} = \frac{\partial \bar{w}}{\partial z}.$$

Theorem 3.14. *Let $U \subseteq \mathbb{C}$ be open, $f\colon U \to \mathbb{C}$, $f = u + iv$. Then f is holomorphic on U if and only if f is C^1-smooth (i.e., u and v are C^1-smooth) and*

$$\frac{\partial f}{\partial \bar{z}} = 0$$

on U (complex version of the Cauchy-Riemann equations).

In this case, $f' = \dfrac{\partial f}{\partial z}$.

Proof. We have

$$\begin{aligned} \frac{\partial f}{\partial \bar{z}} &= \frac{1}{2}(f_x + i f_y) = \frac{1}{2}(u_x + i v_x) + \frac{i}{2}(u_y + i v_y) \\ &= \frac{1}{2}(u_x - v_y) + \frac{i}{2}(v_x + u_y). \end{aligned}$$

So the condition $\dfrac{\partial f}{\partial \bar{z}} = 0$ is equivalent to the Cauchy-Riemann equations. The first part of the statement follows from Theorem 3.8.

In case of holomorphicity we have

$$f' = u_x + iv_x = \frac{1}{i}(u_y + iv_y).$$

Hence

$$f' = \frac{1}{2}(u_x + iv_x) - \frac{i}{2}(u_y + iv_y) = \frac{\partial u}{\partial z} + i\frac{\partial v}{\partial z} = \frac{\partial f}{\partial z}.$$

$\square$

Example 3.15. Let $f(z) = z|z|^2$ for $z \in \mathbb{C}$. Is f anywhere holomorphic in $\mathbb{C}$? Obviously, $f \in C^\infty(\mathbb{C})$, and we have

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial(z^2\bar{z})}{\partial \bar{z}} = z^2.$$

So the Cauchy-Riemann equations are true only for $z = 0$. Hence there is no open set $U \subseteq \mathbb{C}$ where $f|U$ is holomorphic.

Definition 3.16. Let $U \subseteq \mathbb{C}$ be open, $u\colon U \to \mathbb{R}$ be C^2-smooth and harmonic. A harmonic function $v\colon U \to \mathbb{R}$ is called a *harmonic conjugate* of u in U if $f = u + iv$ is holomorphic.

Remark 3.17. (a) It easily follows from the Cauchy-Riemann equations that if a harmonic conjugate exists on a region U, then it is unique up to an additive constant (exercise!).

(b) In general a harmonic conjugate need not exist. If U satisfies a special geometric condition, then *every* harmonic function u on U has a harmonic conjugate. The requirement is that U should have "no holes" and is *simply connected*. This concept will be later discussed in detail in this course.

Example 3.18. (a) Let $u(x, y) = x^2 - y^2$ for $(x, y) \in \mathbb{R}^2$. Then $u \in C^\infty(\mathbb{R}^2)$, $u_x = 2x$, $u_y = -2y$, and $u_{xx} = 2$, $u_{yy} = -2$. Hence $\Delta u = 0$, and so u is harmonic.

Does u have a harmonic conjugate, and how to find it? We use the Cauchy-Riemann equation that a harmonic conjugate v of u must satisfy:

$$v_x = -u_y = 2y, \quad v_y = u_x = 2x.$$

By the first equation we have

$$v(x, y) = \int 2y\, dx + C(y) = 2xy + C(y).$$

Using this, we get
$$v_y = 2x + C'(y) = u_x = 2x.$$
Hence $C'(y) = 0$, and so $C(y) \equiv \text{const.} =: c \in \mathbb{R}$. So $v(x,y) = 2xy + c$. If we set
$$f(\underbrace{x+iy}_{z}) := u(x,y) + iv(x,y) = x^2 - y^2 + 2ixy + c = z^2 + c,$$
then f is indeed holomorphic.

(b) Let $u(x,y) = \frac{1}{2}\log(x^2+y^2)$ for $(x,y) \in U := \mathbb{R}^2 \setminus \{0\}$. Then $u \in C^\infty(\mathbb{R}^2 \setminus \{0\})$,
$$u_x = \frac{x}{x^2+y^2}, \quad u_y = \frac{y}{x^2+y^2}, \quad u_{xx} = \frac{-x^2+y^2}{(x^2+y^2)^2}, \quad u_{yy} = \frac{x^2-y^2}{(x^2+y^2)^2}.$$
Hence $\Delta u = 0$, and so u is harmonic on U.

Does u have a harmonic conjugate v? If so, then
$$v_y = u_x = \frac{x}{x^2+y^2},$$
and so at least locally, where $x \neq 0$,
$$v = \int \frac{x}{x^2+y^2}\,dy + C(x) = \arctan(y/x) + C(x).$$
Then
$$v_x = -\frac{y}{x^2+y^2} + C'(x) = -u_y = -\frac{y}{x^2+y^2},$$
so $C(x) \equiv \text{const.} =: c$, and
$$v(x,y) = \arctan(y/x) + c$$
locally near points (x,y) where $x \neq 0$.

If we introduce polar coordinates, and set $z = x + iy = re^{i\varphi}$, then locally near points (x,y) where $x \neq 0$,
$$v(x,y) = \arctan(y/x) + c = \varphi + c',$$
where c' is locally constant.

If u had a harmonic conjugate, then by continuity necessarily
$$v(x,y) = \varphi + c',$$

for *all* $(x, y) \in U$; but there is *no* well-defined function $(x, y) \mapsto \varphi$ on *all* of U (it exists locally near each point, or on the *slit plane* $\mathbb{C} \setminus (-\infty, 0]$, for example).

The conclusion is that u has no harmonic conjugate *on* U; on a more intuitive level, u has a locally harmonic conjugate (unique up to constant), but it changes globally by the period 2π if we run around 0 counter-clockwise once.

This is an instance of a more general fact: on arbitrary regions U every harmonic function has a local harmonic conjugate v, but v changes by certain additive *periods* if we run around the "holes" of U.

4 Path integrals

Definition 4.1. A *path* is a continuous map $\gamma\colon [a,b] \to \mathbb{C}$ defined on a compact interval $[a,b] \subseteq \mathbb{R}$. The *image* or *trace* of the path is the set $\gamma([a,b])$ denoted by γ^*. The *endpoints* of γ are $\gamma(a)$ and $\gamma(b)$. The path is a *loop* if $\gamma(a) = \gamma(b)$. If $\gamma^* \subseteq U \subseteq \mathbb{C}$, then we say γ is a path *in* U.

The path γ is called *piecewise smooth* if there exists a partition $t_0 = a < t_1 < \cdots < t_n = b$ of $[a,b]$ such that γ is differentiable on each interval $[t_{k-1}, t_k]$, $k \in \{1, \ldots, n\}$, with continuous derivative γ'.

Definition 4.2. Let $h\colon [a,b] \to \mathbb{C}$ be a function. We call h *(Riemann) integrable* if $\operatorname{Re}(h)$ and $\operatorname{Im}(h)$ are (Riemann) integrable on $[a,b]$. If h is integrable, we define

$$\int_a^b h(t)\,dt := \int_a^b (\operatorname{Re} h)(t)\,dt + i\int_a^b (\operatorname{Im} h)(t)\,dt.$$

Proposition 4.3. *Let $[a,b] \subseteq \mathbb{R}$ be compact interval, $f, g, h\colon [a,b] \to \mathbb{C}$ be integrable functions, and $\alpha, \beta \in \mathbb{C}$. Then the following statements are true:*

(a) $\displaystyle\int_a^b (\alpha f + \beta g)(t)\,dt = \alpha\int_a^b f(t)\,dt + \beta\int_a^b g(t)\,dt,$

(b) $\displaystyle\left|\int_a^b h(t)\,dt\right| \leq \int_a^b |h(t)|\,dt,$

(c) *if h has a* primitive *H, i.e., H is differentiable on $[a,b]$ and $H' = h$, then*

$$\int_a^b h(t)\,dt = H(b) - H(a).$$

Proof. The statements (a) and (c) follow from the corresponding statements for integrals of real-valued functions if we split the relevant expressions up into real and imaginary parts.

(b) Assume $h = u + iv$, where $u = \operatorname{Re}(h)$ and $v = \operatorname{Im}(h)$. Define $\alpha := \int_a^b u(t)\,dt$ and $\beta := \int_a^b v(t)\,dt$. Then by the Cauchy-Schwarz inequality we have

$$\left|\int_a^b h(t)\,dt\right|^2 = \alpha^2+\beta^2 = \int_a^b \big(\alpha u(t)+\beta v(t)\big)\,dt$$
$$\le (\alpha^2+\beta^2)^{1/2}\int_a^b \big(u(t)^2+v(t)^2\big)^{1/2}\,dt$$
$$= \left|\int_a^b h(t)\,dt\right|\cdot\int_a^b |h(t)|\,dt.$$

Inequality (b) follows. □

Definition 4.4 (Path integrals)**.** Let $\gamma\colon [a,b]\to\mathbb{C}$ be a piecewise smooth path, and $f\colon \gamma^*\to\mathbb{C}$ be a continuous function. Then we define the *(path) integral of f over γ* (denoted by $\int_\gamma f(z)\,dz$, $\int_\gamma f$, etc.), as

$$\int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\gamma'(t)\,dt.$$

Example 4.5. Let $a\in\mathbb{C}$, $r>0$, and

$$\gamma(t)=a+re^{it},\quad t\in[0,2\pi].$$

This is a positively-oriented parametrization of the circle of radius r centered at a. Let $f(z)=(z-a)^n$, $n\in\mathbb{Z}$. Then $\gamma'(t)=ire^{it}$, and so

$$\int_\gamma (z-a)^n\,dz = \int_0^{2\pi}(re^{it})^n ire^{it}\,dt = ir^{n+1}\int_0^{2\pi} e^{i(n+1)t}\,dt$$
$$=\begin{cases} i\displaystyle\int_0^{2\pi} dt = 2\pi i, & n=-1,\\[2ex] \dfrac{ir^{n+1}}{i(n+1)}e^{i(n+1)t}\Big|_0^{2\pi} = 0, & n\ne -1.\end{cases}$$

If we run through the circle with different speed (e.g., $\gamma(t)=a+re^{2\pi it}$, $t\in[0,1]$), or with different orientation ($\gamma(t)=a+re^{-it}$, $t\in[0,2\pi]$), then the integrals are unchanged or change sign, respectively.

4.6. Reparametrization of paths. Let $\alpha\colon [a,b]\to\mathbb{C}$ and $\beta\colon [c,d]\to\mathbb{C}$ be piecewise smooth paths. Then β is called a *reparametrization* of α if there exists a bijection $\varphi\colon [c,d] \to [a,b]$ such that φ and φ^{-1} are C^1-smooth, $\varphi(c) = a$, $\varphi(d) = b$, and $\beta = \alpha \circ \varphi$.

Integrals do not change under reparametrizations: if β is a reparametrization of α and $f\colon \alpha^* = \beta^* \to \mathbb{C}$ is continous, then

$$\int_\beta f = \int_\alpha f.$$

Indeed,

$$\begin{aligned}\int_\beta f = \int_\beta f(w)\,dw &= \int_c^d f(\beta(s))\beta'(s)\,ds \\ &= \int_c^d f(\alpha(\varphi(s)))\alpha'(\varphi(s))\varphi'(s)\,ds \quad (t = \varphi(s)) \\ &= \int_a^b f(\alpha(t))\alpha'(t)\,dt = \int_\alpha f(z)\,dz = \int_\alpha f.\end{aligned}$$

The paths α and β are called *equivalent* if β is a reparametrization of α (this is indeed an equivalence relation for paths!).

The *oriented interval* $[a,b]$ for $a, b \in \mathbb{C}$ is the path $\gamma\colon [0,1] \to \mathbb{C}$, $\gamma(t) := (1-t)a + tb$, $t \in [0,1]$ (or any equivalent path). In this case, one writes $\int_{[a,b]} f(z)\,dz$ instead of $\int_\gamma f(z)\,dz$.

4.7. Computational rules for path integrals. (a) Let $\gamma\colon [a,b] \to \mathbb{C}$ be a piecewise smooth path, $f, g\colon \gamma^* \to \mathbb{C}$ be continuous, and $c, d \in \mathbb{C}$. Then

$$\int_\gamma \big(cf(z) + dg(z)\big)\,dz = c\int_\gamma f(z)\,dz + d\int_\gamma g(z)\,dz.$$

(b) Suppose γ_1 and γ_2 are piecewise smooth paths such that the endpoint of γ_1 is equal to the initial point of γ_2. Then we may without loss of generality assume that $\gamma_1\colon [0,1] \to \mathbb{C}$, $\gamma_2\colon [1,2] \to \mathbb{C}$ and $\gamma_1(1) = \gamma_2(1)$. Then the *concatenation* of γ_1 and γ_2 is the path $\gamma\colon [0,2] \to \mathbb{C}$ defined as

$$\gamma(t) = \begin{cases} \gamma_1(t), & t \in [0,1], \\ \gamma_2(t), & t \in [1,2]. \end{cases}$$

If $f\colon \gamma_1^* \cup \gamma_2^* \to \mathbb{C}$ is continuous, then

$$\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz.$$

(c) Let $U \subseteq \mathbb{C}$ be open, $\gamma\colon [a,b] \to U$ be a piecewise smooth path, and $f\colon U \to \mathbb{C}$ be a continuous function.

If f has a primitive F, i.e., $F \in H(U)$ and $F' = f$, then

$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a)).$$

In particular, $\displaystyle\int_\gamma f(z)\,dz = 0$ if γ is a loop. Indeed,

$$\frac{d}{dt}F(\gamma(t)) = F'(\gamma(t))\gamma'(t) = f(\gamma(t))\gamma'(t),$$

and so

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt = F(\gamma(t))\Big|_a^b = F(\gamma(b)) - F(\gamma(a)).$$

4.8. Lengths of paths. Let $\gamma\colon [a,b] \to X$ be a path in a metric space (X,d). One defines its *length* $\ell(\gamma) \in [0,\infty]$ as

$$\ell(\gamma) := \sup_{a=t_0<t_1<\dots<t_n=b} \sum_{k=1}^n d(\gamma(t_{k-1}),\gamma(t_k)),$$

where the supremum is taken over all partitions of $[a,b]$. One says that γ is *rectifiable* if $\ell(\gamma) < \infty$.

If $\gamma\colon [a,b] \to \mathbb{C}$ is piecewise smooth path, one can show that γ is rectifiable and

$$\ell(\gamma) = \int_a^b |\gamma'(t)|\,dt. \tag{2}$$

Since we are only interested in piecewise smooth paths in $\mathbb{C}$, we can take (2) as the definition of the length of such a path.

This notion of length has the expected properties (such as invariance under reparametrizations, or additivity under concatenation of paths).

Lemma 4.9. *Let $\gamma\colon [a,b] \to \mathbb{C}$ be a piecewise smooth path, $f\colon \gamma^* \to \mathbb{C}$ be a continuous function, and $M \geq 0$ be such that $|f(z)| \leq M$ for $z \in \gamma^*$. Then*

$$\left|\int_\gamma f(z)\,dz\right| \leq M\ell(\gamma).$$

Proof.

$$\begin{aligned} \left|\int_\gamma f(z)\,dz\right| &= \left|\int_a^b f(\gamma(t))\gamma'(t)\,dt\right| \\ &\le \int_a^b \big|f(\gamma(t))\big| \cdot \big|\gamma'(t)\big|\,dt \qquad \text{(by Prop. 4.3 (b))} \\ &\le M\int_a^b \big|\gamma'(t)\big|\,dt = M\ell(\gamma). \end{aligned}$$

□

5 Power series

Definition 5.1. With a given sequence $\{a_n\}$ in $\mathbb{C}$ we associate another sequence $\{s_n\}$ defined by

$$s_n = a_1 + \cdots + a_n$$

for $n \in \mathbb{N}$. The sequence $\{s_n\}$ is symbolically represented by the symbol $\displaystyle\sum_{n=1}^{\infty} a_n$, called an *infinite series*. The terms s_n are called the *partial sums* of this infinite series. We say that $\displaystyle\sum_{n=1}^{\infty} a_n$ *converges* or *diverges* depending on whether the limit $\displaystyle\lim_{n\to\infty} s_n$ exists or not. In case of convergence, we also denote by $\displaystyle\sum_{n=1}^{\infty} a_n$ the limit $\displaystyle\lim_{n\to\infty} s_n$.

Theorem 5.2. *Let $a_n \in \mathbb{C}$ for $n \in \mathbb{N}$.*

(a) *If $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\lim_{n\to\infty} a_n = 0$.*

(b) *If $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges (in which case we say that $\displaystyle\sum_{n=1}^{\infty} a_n$* converges absolutely*), then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.*

(c) *Let $b_n \geq 0$ for $n \in \mathbb{N}$. If $|a_n| \leq b_n$ for $n \in \mathbb{N}$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges* (comparison test).

(d) *Suppose that $a_n \neq 0$ for $n \in \mathbb{N}$ and $\displaystyle\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$. Then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges* (ratio test).

(e) *Suppose that $\displaystyle\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1$. Then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges* (root test).

Proof. The proofs are similar to the proofs for series of real numbers. We will prove (c) and (e).

(c) Let $s_n = \sum_{k=1}^{n} a_k$ and $t_n = \sum_{k=1}^{n} b_k$ for $n \in \mathbb{N}$. By completeness of $\mathbb{C}$ it is enough to show that $\{s_n\}$ is a Cauchy sequence.

To see this let $\epsilon > 0$ be arbitrary. Since $\{t_n\}$ converges, this sequence is a Cauchy sequence. Hence there exists $N \in \mathbb{N}$ such that $|t_n - t_m| < \epsilon$ whenever $m, n \geq N$.

Suppose $n, m \geq N$. Without loss of generality we may also assume that $n \geq m$. Then

$$\begin{aligned}
|s_n - s_m| &= \left| \sum_{k=1}^{n} a_k - \sum_{k=1}^{m} a_k \right| \\
&= \left| \sum_{k=m+1}^{n} a_k \right| \leq \sum_{k=m+1}^{n} |a_k| \\
&\leq \sum_{k=m+1}^{n} b_k = t_n - t_m < \epsilon.
\end{aligned}$$

So $\{s_n\}$ is indeed a Cauchy sequence.

(e) The number $s^* = \limsup_{n\to\infty} c_n$ for a bounded sequence of real numbers $\{c_n\}$ is characterized by the following properties:

(i) If $s_1 < s^*$, then there are infinitely many $n \in \mathbb{N}$ with $s_1 \leq c_n$.

(ii) If $s^* < s_2$, then there are only finitely many $n \in \mathbb{N}$ with $s_2 \leq c_n$.

Now let $q := \limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1$. Pick s such that $q < s < 1$. Then by (ii) we have

$$s \leq \sqrt[n]{|a_n|}$$

for only finitely many $n \in \mathbb{N}$. Hence there exists $N \in \mathbb{N}$ such that

$$\sqrt[n]{|a_n|} \leq s \quad \text{for all } n \in \mathbb{N},\, n \geq N.$$

This implies that

$$|a_n| \leq s^n \quad \text{for all } n \in \mathbb{N},\, n \geq N.$$

Since $\sum_{n=N}^{\infty} s^n$ converges (geometric series!), the series $\sum_{n=N}^{\infty} a_n$ also converges by the comparison test. Hence $\sum_{n=1}^{\infty} a_n$ converges. $\square$

Example 5.3. (a) For $z \in \mathbb{C}$ the series $\sum_{n=0}^{\infty} z^n$ converges precisely if $|z| < 1$, and we have

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \quad |z| < 1.$$

This easily follows from the fact that

$$\sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1-z}$$

for $z \neq 1$.

(b) The series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges for all $z \in \mathbb{C}$ by the ratio test. Indeed, convergence is clear for $z = 0$. For $z \neq 0$ put $a_n = \frac{z^n}{n!}$ for $n \in \mathbb{N}_0$. Then

$$\begin{aligned}
\limsup_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \limsup_{n\to\infty} \frac{|z|^{n+1}}{(n+1)!} \cdot \frac{n!}{|z|^n} \\
&= \lim_{n\to\infty} \frac{|z|}{n+1} = 0 < 1.
\end{aligned}$$

Definition 5.4 (Power series)**.** A series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

with fixed $z_0 \in \mathbb{C}$ and $a_n \in \mathbb{C}$ for $n \in \mathbb{N}_0$, is called a *(complex) power series (in z centered at z_0)*.

The convergence of the series depends on z.

Theorem 5.5. *Let* $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *be a power series and define*

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}} \in [0,\infty]. \tag{3}$$

Then the power series $\left\{ \begin{array}{c} \textit{converges} \\ \textit{diverges} \end{array} \right\}$ *for all* $z \in \mathbb{C}$ *with* $\left\{ \begin{array}{c} |z-z_0| < R \\ |z-z_0| > R \end{array} \right\}$.

In (3) we use the conventions $1/0 = \infty$ and $1/\infty = 0$. For a point $z \in \mathbb{C}$ with $|z - z_0| = R$ both convergence or divergence may happen.

The value R is called the *radius of convergence* of the power series.

Proof. We will only consider the case $R \in (0,\infty)$. The cases $R = 0$ and $R = \infty$ are similar (and easier).

Let $z \in \mathbb{C}$ with $|z-z_0| > R$ be arbitrary. We want to show that the series diverges at z. Pick $R' \in \mathbb{R}$ with $R < R' < |z - z_0|$. Then

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{R} > \frac{1}{R'},$$

and so

$$\sqrt[n]{|a_n|} > \frac{1}{R'} \quad \text{for infinitely many } n \in \mathbb{N}.$$

Hence

$$|a_n| \cdot |z - z_0|^n > \frac{|z-z_0|^n}{(R')^n} \geq 1 \quad \text{for infinitely many } n \in \mathbb{N}.$$

This means that the terms in $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ do *not* converge to 0, and so the series diverges.

Let $z \in \mathbb{C}$ with $|z - z_0| < R$ be arbitrary. Then

$$\begin{aligned} \limsup_{n\to\infty} \sqrt[n]{|a_n| \cdot |z-z_0|^n} &= \limsup_{n\to\infty} |z-z_0| \sqrt[n]{|a_n|} \\ &= |z-z_0| \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \frac{|z-z_0|}{R} < 1, \end{aligned}$$

and so the power series converges by the root test. □

Remark 5.6. On can show that the radius of convergence R of a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is given by

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \in [0, \infty]$$

if the last limit exists (possibly as an improper limit with value ∞).

Lemma 5.7. *If $\sum_{n=0}^{\infty} a_n(a-z_0)^n$ is a complex power series, then the series*

$$\sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (z-z_0)^n$$

(obtained by "term-by-term differentiation") has the same radius of convergence.

Proof. Note that $\sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}$ converges if and only if

$$(z-z_0) \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1} = \sum_{n=1}^{\infty} n a_n (z-z_0)^n$$

converges. So by Theorem 5.5 it suffices to show that

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \limsup_{n\to\infty} \sqrt[n]{n|a_n|}.$$

This follows from

$$\begin{aligned} \limsup_{n\to\infty} \sqrt[n]{|a_n|} &\le \limsup_{n\to\infty} \sqrt[n]{n|a_n|} \\ &\le \underbrace{\limsup_{n\to\infty} \sqrt[n]{n}}_{=1} \cdot \limsup_{n\to\infty} \sqrt[n]{|a_n|} \\ &= \limsup_{n\to\infty} \sqrt[n]{|a_n|}. \end{aligned}$$

$\square$

Lemma 5.8. (a) *For all $w, z \in \mathbb{C}$, $w \neq z$, $n \in \mathbb{N}$, we have*

$$\frac{w^n - z^n}{w - z} - nz^{n-1} = \left(\sum_{k=1}^{n-1} kz^{k-1}w^{n-k-1}\right)(w - z).$$

(b) *If $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ is a power series with radius of convergence R and $0 \leq r < R$, then $\sum_{n=1}^{\infty} n^2|a_n|r^n$ converges.*

Proof. (a)

$$\begin{aligned}
(w - z)\sum_{k=1}^{n-1} kz^{k-1}w^{n-k-1} &= \sum_{k=1}^{n-1} kz^{k-1}w^{n-k} - \sum_{k=1}^{n-1} kz^k w^{n-k-1} \\
&= \sum_{k=0}^{n-2}(k+1)z^k w^{n-k-1} - \sum_{k=1}^{n-1} kz^k w^{n-k-1} \\
&= w^{n-1} - (n-1)z^{n-1} + \sum_{k=1}^{n-2} z^k w^{n-k-1} \\
&= \sum_{k=0}^{n-1} z^k w^{n-k-1} - nz^{n-1} \\
&= \frac{w^n - z^n}{w - z} - nz^{n-1}.
\end{aligned}$$

In the last step we used that

$$\begin{aligned}
\frac{w^n - z^n}{w - z} &= w^{n-1} + w^{n-2}z + \cdots + wz^{n-2} + z^{n-1} \\
&= \sum_{k=0}^{n-1} z^k w^{n-k-1}.
\end{aligned}$$

(b) We apply the root test:

$$\limsup_{n\to\infty} \sqrt[n]{n^2|a_n|r^n} \leq r \limsup_{n\to\infty} \sqrt[n]{|a_n|} \cdot \underbrace{\limsup_{n\to\infty}(\sqrt[n]{n})^2}_{=1} = \frac{r}{R} < 1.$$

$\square$

Theorem 5.9. *Suppose that* $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *is a complex power series with radius of convergence* $R > 0$. *Let* $D := B(z_0, R)$ *(*$= \mathbb{C}$ *if* $R = \infty$*) and*

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n \quad \text{for } z \in D.$$

Then f *is holomorphic on* D*, and*

$$f'(z) = \sum_{n=0}^{\infty} (n+1)a_{n+1}(z-z_0)^n \quad \text{for } z \in D. \tag{4}$$

Note that the last power series converges for $z \in D$ by Lemma 5.7.

The statement says that a function represented by a power series is holomorphic in its disk of convergence, and the derivative can be obtained by term-by-term differentiation of the power series.

Proof. Without loss of generality $z_0 = 0$. Fix $z \in D$ and choose $r > 0$ such that $|z| < r < R$. We denote by $g(z)$ the value of the power series in (4). We have to show that

$$\lim_{w \to z} \left(\frac{f(w) - f(z)}{w - z} - g(z) \right) = 0. \tag{5}$$

To see this consider $w \in D$ close to z, $w \neq z$. Then $|w| < r$, and so

$$\begin{aligned} \left| \frac{f(w)-f(z)}{w-z} - g(z) \right| &= \left| \sum_{n=1}^{\infty} a_n \left(\frac{w^n - z^n}{w-z} - nz^{n-1} \right) \right| \\ &\leq \sum_{n=1}^{\infty} |a_n| \cdot \left| \frac{w^n - z^n}{w-z} - nz^{n-1} \right| \\ &= |w-z| \sum_{n=1}^{\infty} \left| a_n \sum_{k=1}^{n-1} kz^{k-1}w^{n-k-1} \right| \quad \text{(by Lem. 5.8 (a))} \\ &\leq |w-z| \sum_{n=1}^{\infty} \left(|a_n| \sum_{k=1}^{n-1} kr^{n-2} \right) \\ &= |w-z| \sum_{n=1}^{\infty} \frac{n(n-1)}{2} |a_n| r^{n-2} \\ &\leq \frac{|w-z|}{r^2} \sum_{n=1}^{\infty} n^2 |a_n| r^n \leq C|w-z| \quad \text{(by Lem. 5.8 (b))}, \end{aligned}$$

where $C > 0$ is independent of w. Letting $w \to z$, we see that (5) holds. $\square$

Corollary 5.10. *A function represented by a power series*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

has derivatives of arbitrarily high order:

$$\begin{aligned} f^{(k)}(z) &= \sum_{n=k}^{\infty} n(n-1)\dots(n-k+1) a_n (z - z_0)^{n-k} \\ &= \sum_{n=0}^{\infty} (n+k)\dots(n+1) a_{n+k} (z - z_0)^n, \quad k \in \mathbb{N}_0. \end{aligned}$$

In particular,

$$f^{(k)}(z_0) = k! a_k,$$

and so

$$a_k = \frac{f^{(k)}(z_0)}{k!}.$$

The last equation says that the coefficients of a power series equal the Taylor coefficients of the functions that it represents.

Theorem 5.11. *The exponential function has the power series representation*

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \text{for } z \in \mathbb{C}.$$

Proof. We know that this power series converges for all $z \in \mathbb{C}$ (see Example 5.3 (b)). Let $E\colon \mathbb{C} \to \mathbb{C}$ be the function represented by the power series. Then by Theorem 5.9 the function E is holomorphic in $\mathbb{C}$ and

$$E'(z) = \sum_{n=1}^{\infty} n \frac{z^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = E(z)$$

for all $z \in \mathbb{C}$. We know that the exponential function $z \mapsto e^z$ is also holmorphic on $\mathbb{C}$, and $\frac{d}{dz} e^z = e^z$. Hence

$$\frac{d}{dz}(E(z)e^{-z}) = E'(z)e^{-z} - E(z)e^{-z} = E(z)e^{-z} - E(z)e^{-z} = 0,$$

and so by Theorem 3.5 the function $z \mapsto E(z)e^{-z}$ is constant on $\mathbb{C}$. Since

$$E(0)e^{-0} = 1 \cdot 1 = 1$$

it follows that

$$E(z)e^{-z} \equiv 1.$$

Hence

$$E(z) = E(z)e^0 = E(z)e^{-z}e^z = 1 \cdot e^z = e^z$$

for all $z \in \mathbb{C}$. □

Definition 5.12. Suppose that $F\colon A \to \mathbb{C}$, and $F_n\colon A \to \mathbb{C}$ for $n \in \mathbb{N}$ are functions defined on $A \subseteq \mathbb{C}$. We say that the function sequence $\{F_n\}$ *converges uniformly on A to F* if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$ and for all $z \in A$,

$$|F_n(z) - F(z)| < \epsilon.$$

This implies *pointwise convergence*, namely $\lim_{n\to\infty} F_n(z) = F(z)$ for all $z \in A$, or equivalently: for all $z \in A$ and for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$,

$$|F_n(z) - F(z)| < \epsilon.$$

We say that a function series $\sum_{n=1}^{\infty} f_n$ of functions $f_n\colon A \to \mathbb{C}$ *converges uniformly on A* if the sequence of partial sum $\{F_n\}$, where $F_n = \sum_{k=1}^{n} f_k$ for $n \in \mathbb{N}$, converges uniformly on A to some limit function F.

Theorem 5.13 (Weierstrass M-test)**.** *Suppose that $f_n\colon A \to \mathbb{C}$ for $n \in \mathbb{N}$ are functions defined on $A \subseteq \mathbb{C}$, and that there exist numbers $M_n \geq 0$ such that*

$$|f_n(z)| \leq M_n \quad \text{for all } z \in A \text{ and all } n \in \mathbb{N}.$$

If the series $\sum_{n=1}^{\infty} M_n$ converges, then the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on A.

The symbol "M" in "Weierstrass M-test" stands for "majorization".

Proof. By the comparision test, the series $\sum_{n=1}^{\infty} f_n(z)$ converges for all $z \in A$. Define $F(z) := \sum_{n=1}^{\infty} f_n(z)$ for $z \in A$, and $L := \sum_{n=1}^{\infty} M_n$.

We claim that $\sum_{n=1}^{\infty} f_n$ converges to F uniformly on A. To see this, let $\epsilon > 0$ be arbitrary. Note that

$$\lim_{n\to\infty} \sum_{k=n+1}^{\infty} M_k = \lim_{n\to\infty} \left(L - \sum_{k=1}^{n} M_k \right) = 0$$

("tails" of a convergent series tend to 0).

Hence there exists $N \in \mathbb{N}$ such that $\sum_{k=N+1}^{\infty} M_k \leq \epsilon$. Now if $z \in A$ and $n \in \mathbb{N}$ with $n \geq N$ are arbitrary,

$$\begin{aligned} \left| F(z) - \sum_{k=1}^{n} f_k(z) \right| &= \left| \sum_{k=1}^{\infty} f_k(z) - \sum_{k=1}^{n} f_k(z) \right| \\ &= \left| \sum_{k=n+1}^{\infty} f_k(z) \right| \\ &\leq \sum_{k=n+1}^{\infty} |f_k(z)| \\ &\leq \sum_{k=n+1}^{\infty} M_k \leq \epsilon. \end{aligned}$$

The claim follows. □

Theorem 5.14. *Let $F\colon A \to \mathbb{C}$, and $F_n\colon A \to \mathbb{C}$ for $n \in \mathbb{N}$ be functions defined on $A \subseteq \mathbb{C}$.*

Suppose that the functions F_n for $n \in \mathbb{N}$ are continuous on A, and that the function sequence $\{F_n\}$ converges uniformly to F on A. Then F is continuous on A (uniform limits of continuous functions are continuous).

Moreover, if $\gamma\colon [a, b] \to A$ is a piecewise smooth path, then

$$\lim_{n\to\infty} \int_\gamma F_n(z)\, dz = \int_\gamma F(z)\, dz.$$

Since $F(z) = \lim_{n\to\infty} F_n(z)$ for $z \in A \supset \gamma^*$, the last statement can also be written as

$$\lim_{n\to\infty} \int_\gamma F_n(z)\,dz = \int_\gamma \left(\lim_{n\to\infty} F_n(z) \right) dz.$$

More informally, this says that in the presence of uniform convergence one can interchange the limit and path integration.

Proof. The first part of the statement is well-known; we will only prove the second part.

Let $\epsilon > 0$ be arbitrary. Since $\{F_n\}$ converges to F uniformly on A, there exists $N \in \mathbb{N}$ such that

$$|F_n(z) - F(z)| \le \frac{\epsilon}{\ell(\gamma) + 1},$$

whenever $z \in A$, and $n \in \mathbb{N}$, $n \ge N$. Here $\ell(\gamma)$ is the length of γ.

Hence for all $n \ge N$ we have

$$\begin{aligned} \left| \int_\gamma F_n(z)\,dz - \int_\gamma F(z)\,dz \right| &= \left| \int_\gamma (F_n(z) - F(z))\,dz \right| \\ &\le \frac{\epsilon}{\ell(\gamma)+1} \ell(\gamma) \quad \text{(by Lem. 4.9)} \\ &\le \epsilon. \end{aligned}$$

The claim follows. □

Remark 5.15. We will use the last statement also for function series: if $f_n \colon A \to \mathbb{C}$ for $n \in \mathbb{N}$ are continuous functions defined on $A \subseteq \mathbb{C}$, and $\sum_{n=1}^\infty f_n$ converges uniformly on A, then

$$\begin{aligned} \int_\gamma \left(\sum_{k=1}^\infty f_k(z) \right) dz &= \int_\gamma \left(\lim_{n\to\infty} \sum_{k=1}^n f_k(z) \right) dz \\ &= \lim_{n\to\infty} \int_\gamma \left(\sum_{k=1}^n f_k(z) \right) dz \\ &= \lim_{n\to\infty} \sum_{k=1}^n \int_\gamma f_k(z)\,dz \\ &= \sum_{k=1}^\infty \int_\gamma f_k(z)\,dz. \end{aligned}$$

So a uniformly convergent series of (continuous) functions can be integrated term-by-term.

6 Local Cauchy theorems

Remark 6.1. Cauchy's Integral Theorem says that if f is a holomorphic function on a region $\Omega \subseteq \mathbb{C}$ and γ is a loop in Ω such that all points "surrounded" by γ also belong to Ω, then $\int_\gamma f = 0$.

Since it is requires some effort to make the phrase "points surrounded by γ" mathematically precise, we will prove a preliminary version of this statement first.

6.2. Convex sets. A set $K \subseteq \mathbb{C}$ is called *convex* if for all $z, w \in K$ we have $[z, w] \subseteq K$. One can show that a set is convex if and only if for all $z_1, \ldots, z_n \in K$, $n \in \mathbb{N}$, and all $\lambda_1, \ldots, \lambda_n \in [0, 1]$ with $\lambda_1 + \cdots + \lambda_n = 1$ the *convex combination* $\lambda_1 z_1 + \cdots + \lambda_n z_n$ also belongs to K. If $M \subseteq \mathbb{C}$ is any set, then the *convex hull* $\operatorname{co}(M)$ is the smallest convex set containing M; it consists of all convex combinations of points in M.

A *closed triangle* Δ is the convex hull of three points $z_1, z_2, z_3 \in \mathbb{C}$ called the *vertices* of $\mathbb{C}$; so

$$\Delta = \operatorname{co}(\{z_1, z_2, z_3\}).$$

We say that Δ is *oriented* if we specify a cyclic order of its vertices z_1, z_2, z_3. We then write $\Delta = \Delta(z_1, z_2, z_3)$ to indicate the vertices and their cyclic order. The *oriented boundary* of $\Delta = \Delta(z_1, z_2, z_3)$ is given by

$$\partial\Delta := [z_1, z_2] \cup [z_2, z_3] \cup [z_3, z_1],$$

where the edges of Δ are traversed according to the cyclic order of the vertices. We write $\int_{\partial\Delta} f$ for the corresponding path integral of a function f defined on $\partial\Delta$.

Theorem 6.3 (Goursat's Lemma)**.** *Let $\Omega \subseteq \mathbb{C}$ be an open set, $p \in \Omega$, and $f\colon \Omega \to \mathbb{C}$ be a continuous function that is holomorphic on $\Omega \setminus \{p\}$. If Δ is a closed oriented triangle contained in Ω, then*

$$\int_{\partial\Delta} f = 0.$$

Proof. Let $a, b, c \in \Omega$ be the vertices of Δ cyclically ordered according to the orientation of Δ. Then $\Delta = \Delta(a, b, c)$.

1. We first assume that $p \notin \Delta$. We will inductively define a nested sequence of closed oriented triangles Δ_n, $n \in \mathbb{N}_0$, in Ω. Let $\Delta_0 := \Delta$. We define $I := \int_{\partial\Delta} f$, and denote by L the length of $\partial\Delta$.

Let $\left\{\begin{array}{c} a' \\ b' \\ c' \end{array}\right\}$ be the midpoint of the interval $\left\{\begin{array}{c} [b,c] \\ [c,a] \\ [a,c] \end{array}\right\}$, and let $\widetilde{\Delta}_1 := \Delta(a, c', b')$, $\widetilde{\Delta}_2 := \Delta(b, a', c')$, $\widetilde{\Delta}_3 := \Delta(c, b', a')$, $\widetilde{\Delta}_4 := \Delta(a', b', c')$. Then

$$I = \int_{\partial\Delta} f = \sum_{k=1}^{4} \int_{\partial\widetilde{\Delta}_k} f,$$

and so

$$|I| = \left| \sum_{k=1}^{4} \int_{\partial\widetilde{\Delta}_k} f \right| \leq \sum_{k=1}^{4} \left| \int_{\partial\widetilde{\Delta}_k} f \right|.$$

Hence there exists $j \in \{1, 2, 3, 4\}$ such that

$$\left| \int_{\partial\widetilde{\Delta}_j} f \right| \geq \frac{1}{4} |I|.$$

Define $\Delta_1 := \widetilde{\Delta}_j$. Note that

$$\ell(\partial\Delta_1) = \frac{1}{2}\ell(\partial\Delta) = \frac{1}{2}L \text{ and } \operatorname{diam}(\Delta_1) = \frac{1}{2}\operatorname{diam}(\Delta) \leq \frac{1}{2}L.$$

Now we apply the same construction to Δ_1 to obtain another closed oriented triangle Δ_2, then to Δ_2 to get Δ_3, etc.

In this way, we get a nested sequence of closed oriented triangles $\Delta_0 \supseteq \Delta_1 \supseteq \Delta_2 \supseteq \dots$ in Ω such that

$$\left| \int_{\partial\Delta_n} f \right| \geq \frac{1}{4^n} |I|$$

for $n \in \mathbb{N}_0$. Moreover, we also have

$$\ell(\partial\Delta_n) = \frac{1}{2^n}L \text{ and } \operatorname{diam}(\Delta_n) \leq \frac{1}{2^n}L$$

for $n \in \mathbb{N}_0$.

If $K_0 \supseteq K_1 \supseteq K_2 \supseteq \dots$ is a nested sequence of compact sets with $\operatorname{diam}(K_n) \to 0$ as $n \to \infty$, then $\bigcap_{n\in\mathbb{N}_0} K_n$ consists of exactly one point (exercise!). This implies that there exists a unique point $z_0 \in \bigcap_{n\in\mathbb{N}_0} \Delta_n \subseteq \Delta \subseteq \Omega \setminus \{p\}$. Since $z_0 \in \Omega$ and $z_0 \neq p$, the function f is differentiable at z_0.

Now let $\epsilon > 0$ be arbitrary. Then there exists $\delta > 0$ such that $B(z_0, \delta) \subseteq \Omega$ and

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| \leq \epsilon, \tag{6}$$

whenever $0 < |z - z_0| < \delta$. We can find $n \in \mathbb{N}$ such that

$$\operatorname{diam}(\Delta_n) \leq \frac{1}{2^n} L < \delta.$$

Then

$$\Delta_n \subseteq \overline{B}(z_0, L/2^n) \subseteq B(z_0, \delta) \subseteq \Omega,$$

and so by (6) we have

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| \leq \epsilon |z - z_0| \tag{7}$$

whenever $z \in \Delta_n$.

The functions $z \mapsto f(z_0)$ and $z \mapsto f'(z_0)(z - z_0)$ have the primitives $z \mapsto z f(z_0)$ and $z \mapsto \frac{1}{2} f'(z_0)(z - z_0)^2$, respectively. Hence by 4.7 (c) we have

$$\int_{\partial \Delta_n} f(z_0)\, dz = 0 = \int_{\partial \Delta_n} f'(z_0)(z - z_0)\, dz.$$

It follows from (7) that

$$\begin{aligned}
\left| \int_{\partial \Delta_n} f(z)\, dz \right| &= \left| \int_{\partial \Delta_n} (f(z) - f(z_0) - f'(z_0)(z - z_0))\, dz \right| \\
&\leq \ell(\partial \Delta_n) \sup_{z \in \partial \Delta_n} |f(z) - f(z_0) - f'(z_0)(z - z_0)| \\
&\leq \frac{1}{2^n} L \sup_{z \in \partial \Delta_n} \epsilon |z - z_0| \\
&\leq \epsilon \frac{1}{2^n} L \operatorname{diam}(\Delta) \leq \epsilon L \frac{1}{2^n} \cdot L \frac{1}{2^n} = \epsilon L^2 \frac{1}{4^n}.
\end{aligned}$$

This implies

$$|I| \leq 4^n \left| \int_{\partial \Delta_n} f(z)\, dz \right| \leq 4^n \epsilon L^2 \frac{1}{4^n} = \epsilon L^2.$$

Since $\epsilon > 0$ was arbitrary, we must have $|I| = 0$, and so $I = \displaystyle\int_{\partial \Delta} f = 0$ as claimed.

2. We now assume that p coincides with one of the vertices a, b, c of Δ, say $p = a$. Without loss of generality, the vertices a, b, c are distinct (otherwise, it is easy to see that the claim trivially holds). Then we may choose points $b' \in [a, b]$ and $c' \in [a, c]$ close to, but distinct from $a = p$.

Now let $\Delta_1 = \Delta(b', b, c)$, $\Delta_2 = \Delta(c, c', b')$, and $\Delta_3 = \Delta(a, b', c')$. Then

$$\int_{\partial\Delta} f = \int_{\partial\Delta_1} f + \int_{\partial\Delta_2} f + \int_{\partial\Delta_3} f.$$

Moreover, $\Delta_1, \Delta_2 \subseteq \Omega \setminus \{p\}$, and so

$$\int_{\partial\Delta_1} f = 0 = \int_{\partial\Delta_2} f$$

by the first part of the proof. Hence

$$\int_{\partial\Delta} f = \int_{\partial\Delta_3} f,$$

and so

$$\left| \int_{\partial\Delta} f \right| \leq \ell(\partial\Delta_3) \sup_{z \in \partial\Delta_3} |f(z)|.$$

Since f is continuous at p, this function is bounded in a neighborhood of p. Choosing b' and c' close enough to p, we can make $\ell(\partial\Delta_3)$ as small as we want, while $\sup_{z \in \partial\Delta_3} |f(z)|$ stays uniformly bounded. It follows that

$$\left| \int_{\partial\Delta} f \right| \leq \epsilon$$

for all $\epsilon > 0$, which implies $\displaystyle\int_{\partial\Delta} f = 0$ as claimed.

3. Finally, we prove the claim if $p \in \Delta$, but where p is not necessarily a vertex of Δ. We cut Δ into three triangles so that p becomes a vertex in each of them; namely, we define $\Delta_1 = \Delta(a, b, p)$, $\Delta_2 = \Delta(b, c, p)$, and $\Delta_3 = \Delta(c, a, p)$. Then

$$\int_{\partial\Delta} f = \int_{\partial\Delta_1} f + \int_{\partial\Delta_2} f + \int_{\partial\Delta_3} f,$$

and by the second part of the proof

$$\int_{\partial\Delta_1} f = \int_{\partial\Delta_2} f = \int_{\partial\Delta_3} f = 0.$$

It follows that $\displaystyle\int_{\partial\Delta} f = 0$.

The proof is complete. $\square$

Corollary 6.4 (Cauchy's Integral Theorem for convex sets)**.** *Suppose $\Omega \subseteq \mathbb{C}$ is an open convex set, $p \in \Omega$, and $f\colon \Omega \to \mathbb{C}$ is a continuous function that is holomorphic on $\Omega \setminus \{p\}$. Then f has a primitive in Ω, i.e., there exists $F \in H(\Omega)$ such that $F' = f$.*

In particular,
$$\int_\gamma f = 0,$$
whenever γ is a piecewise smooth loop in Ω.

Proof. Fix $a \in \Omega$, and define
$$F(w) = \int_{[a,w]} f(z)\, dz$$
for $w \in \Omega$. Since Ω is convex, and so $[a,w] \subseteq \Omega$, and since f is continuous on Ω, the function F is well defined.

Let $w, w_0 \in \Omega$ be arbitrary. Then the triangle $\Delta = \Delta(a, w, w_0)$ lies in Ω. So it follows from Goursat's lemma that
$$\int_{\partial\Delta} f = \int_{[a,w]} f + \int_{[w,w_0]} f + \int_{[w_0,a]} f = 0.$$
Hence
$$F(w) - F(w_0) = \int_{[a,w]} f + \int_{[w_0,a]} f = -\int_{[w,w_0]} f = \int_{[w_0,w]} f.$$
So for $w \neq w_0$ we get
$$\begin{aligned}
\left|\frac{F(w) - F(w_0)}{w - w_0} - f(w_0)\right| &= \\
\left|\frac{1}{w - w_0}\int_{[w_0,w]} f(z)\, dz - \frac{1}{w - w_0}\int_{[w_0,w]} f(w_0)\, dz\right| & \\
&= \frac{1}{|w - w_0|}\left|\int_{[w_0,w]} (f(z) - f(w_0))\, dz\right| \\
&\leq \sup_{z \in [w_0,w]} |f(z) - f(w_0)| \quad \text{(by Lem. 4.9)} \\
&\to 0 \quad \text{as } w \to w_0.
\end{aligned}$$

The last relation follows from the continuity of f at w_0.

So F is differentiable at w_0, and $F'(w_0) = f(w_0)$. Since w_0 was arbitrary, we conclude that $F \in H(\Omega)$ and $F' = f$.

The statement about path integrals now follows from 4.7 (c). □

Lemma 6.5. *Let $a \in \mathbb{C}$, $r > 0$, $\gamma(t) = a + re^{it}$ for $t \in [0, 2\pi]$, and $z_0 \in B(a,r)$. Then*

$$\int_\gamma \frac{dz}{z - z_0} = 2\pi i.$$

Proof. It is not hard, but tedious to evaluate this integral by reducing it to integrals over real-valued functions and applying a trigonometric substitution. We prefer to show the claim by using methods from complex analysis.

Note that for $z \in \mathbb{C}$ with $|z - a| = r$ and $z_0 \in B(a,r)$ we have

$$\frac{1}{z - z_0} = \frac{1}{(z-a) - (z_0 - a)} = \frac{1}{z-a} \cdot \frac{1}{1 - \left(\dfrac{z_0 - a}{z - a}\right)}$$

$$= \frac{1}{z - z_0} \sum_{n=0}^{\infty} \left(\frac{z_0 - a}{z - a}\right)^n = \sum_{n=0}^{\infty} \frac{(z_0 - a)^n}{(z-a)^{n+1}}.$$

This sum converges uniformly in z on the circle $\gamma^* = \{z \in \mathbb{C} : |z - a| = r\}$ (to see this, use the Weierstrass M-test with

$$M_n := \frac{1}{r} \Bigg(\underbrace{\frac{|z_0 - a|}{r}}_{<1}\Bigg)^n$$

for $n \in \mathbb{N}_0$).

Hence by Remark 5.15, we can integrate the infinite series term-by-term and obtain

$$\int_\gamma \frac{dz}{z - z_0} = \sum_{n=0}^{\infty} \int_\gamma \frac{(z_0 - a)^n}{(z-a)^{n+1}}\, dz.$$

Now by Example 4.5 we have

$$\int_\gamma \frac{dz}{(z-a)^{n+1}} = \int_\gamma (z-a)^{-n-1}\, dz = \begin{cases} 2\pi i & \text{for } n = 0, \\ 0 & \text{for } n \neq 0. \end{cases}$$

Hence

$$\int_\gamma \frac{dz}{z - z_0} = 2\pi i \cdot (z_0 - a)^0 = 2\pi i.$$

□

Corollary 6.6 (Cauchy's Integral Formula. Version I)**.** *Let $U \subseteq \mathbb{C}$ be an open set, $a \in U$, $r > 0$, $f \in H(U)$, and $\gamma(t) = a + re^{it}$ for $t \in [0, 2\pi]$. If $\overline{B}(a, r) \subseteq U$, then*

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz$$

for all $z_0 \in B(a, r)$.

Proof. Suppose $\overline{B}(a, r) \subseteq U$, and let $z_0 \in B(a, r)$. Note that $\gamma^* \subseteq U$, and $z_0 \notin \gamma^*$. So $z \mapsto \dfrac{f(z) - f(z_0)}{z - z_0}$ is continuous on γ^* and we can integrate this function over γ^*.

Since $\overline{B}(a, r) \subseteq U$, and U is open, we can find a small number $\delta > 0$ such that for $R := r + \delta$ we have

$$\overline{B}(a, r) \subseteq \Omega := B(a, R) \subseteq U$$

(exercise!). Note that Ω is an open and convex set. Moreover, the function $g \colon \Omega \to \mathbb{C}$ defined by

$$g(z) = \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0} & \text{for } z \in \Omega \setminus \{z_0\}, \\[2ex] f'(z_0) & \text{for } z = z_0, \end{cases}$$

is continuous on Ω and holomorphic on $\Omega \setminus \{z_0\}$. Since $\gamma^* \subseteq \Omega$, Corollary 6.4 implies that

$$\begin{aligned} \int_\gamma \frac{f(z)}{z - z_0}\, dz - 2\pi i f(z_0) = \\ \int_\gamma \frac{f(z)}{z - z_0}\, dz - f(z_0) \int_\gamma \frac{dz}{z - z_0} \quad &\text{(by Lem. 6.5)} \\ = \int_\gamma \frac{f(z) - f(z_0)}{z - z_0}\, dz& \\ = \int_\gamma g(z)\, dz = 0.& \end{aligned}$$

The statement follows. □

7 Power series representations of holomorphic functions

Theorem 7.1. *Let $\Omega \subseteq \mathbb{C}$ be open, and $f \in H(\Omega)$. Then f has a local representation as a power series at every point of Ω; more precisely, if $z_0 \in \Omega$ is arbitrary, and $r > 0$ is such that $\overline{B}(z_0, r) \subseteq \Omega$, then there exists a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ that converges on $B(z_0, r)$ and satisfies*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all $z \in B(z_0, r)$.

Proof. Assume that $\overline{B}(z_0, r) \subseteq \Omega$, fix $z \in B(z_0, r)$, and let $\gamma(t) := z_0 + re^{it}$ for $t \in [0, 2\pi]$. Then by Cauchy's Integral Formula (Corollary 6.6) we have

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z}\, dw.$$

Now for $w \in \partial B(z_0, r)$ we have

$$\frac{1}{w - z} = \frac{1}{(w - z_0) - (z - z_0)} = \frac{1}{(w - z_0)} \frac{1}{1 - \left(\dfrac{z - z_0}{w - z_0}\right)}$$
$$= \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(w - z_0)^{n+1}}.$$

Hence

$$\frac{f(w)}{w - z} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(w - z_0)^{n+1}} f(w) \tag{8}$$

for all $w \in \partial B(z_0, r)$.

The series in (8) (for fixed z) converges uniformly for $w \in \partial B(z_0, r)$ by the Weierstrass M-test; indeed, since $\partial B(z_0, r)$ is a compact set and $|f|$ is continuous on $\partial B(z_0, r)$, there exists $K \geq 0$ such that

$$|f(w)| \leq K \quad \text{for } w \in \partial B(z_0, r).$$

So for $n \in \mathbb{N}_0$ we have

$$\left|\frac{(z-z_0)^n}{(w-z_0)^{n+1}} f(w)\right| \leq \frac{|z-z_0|^n}{r^{n+1}} K =: M_n,$$

and

$$\sum_{n=0}^{\infty} M_n = \frac{K}{r} \sum_{n=0}^{\infty} \frac{|z-z_0|^n}{r^n} = \frac{K}{r} \frac{1}{1-\dfrac{|z-z_0|}{r}} = \frac{K}{r-|z-z_0|} < \infty.$$

Integrating the series in (8) term-by-term, we obtain

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z}\, dw \\ &= \frac{1}{2\pi i} \int_\gamma \left(\sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(w-z_0)^{n+1}} f(w)\right) dw \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_\gamma \frac{(z-z_0)^n}{(w-z_0)^{n+1}} f(w)\, dw\right) \\ &= \sum_{n=0}^{\infty} a_n (z-z_0)^n, \end{aligned} \tag{9}$$

where

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-z_0)^{n+1}}\, dw \tag{10}$$

for $n \in \mathbb{N}_0$.

Since a_n is independent of z, and $z \in B(z_0, r)$ was arbitrary, we see that the series in (9) converges for all $z \in B(z_0, r)$ and represents f. The claim follows. □

Corollary 7.2. *Let $\Omega \subseteq \mathbb{C}$ be open, and $f \in H(\Omega)$. Then $f' \in H(\Omega)$.*

This corollary establishes the fact used in Section 3.

Proof. By Theorem 7.1 the function f has a local power series representation; so f' can also be locally represented by a power series and $f' \in H(\Omega)$ as follows from Theorem 5.9. □

Corollary 7.3. *Let $\Omega \subseteq \mathbb{C}$ be open, and $f \in H(\Omega)$. Then f has derivatives $f^{(n)}$ of arbitrary order $n \in \mathbb{N}$, and $f^{(n)} \in H(\Omega)$.*

Holomorphic functions are indefinitely differentiable!

Corollary 7.4. *Let $\Omega \subseteq \mathbb{C}$ be open, and $f \in H(\Omega)$. Then at each point $z_0 \in \Omega$, the function f is represented by its Taylor series, and so*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n.$$

This is valid for all $z \in \mathbb{C}$ with $|z - z_0| < \operatorname{dist}(z, \partial\Omega)$; in particular, if R is the radius of convergence of the power series, then $R \geq \operatorname{dist}(z_0, \partial\Omega)$.

In other words, the Taylor series of f converges and represents the function in the largest open disk centered at z_0 that is contained in Ω; if $\partial\Omega = \emptyset$ and so $\Omega = \mathbb{C}$, this means that this Taylor series converges and represents the function for *all* $z \in \mathbb{C}$.

Proof. By Theorem 7.1 the function f can locally be represented by a power series centered at z_0. By Corollary 5.10 the coefficients of this power series are uniquely determined, because they are the Taylor coefficients of f at z_0. Since each point $z \in \Omega$ with $|z - z_0| < \operatorname{dist}(z_0, \partial\Omega)$ lies in a disk $B(z_0, r)$ with $\overline{B}(z_0, r) \subseteq \Omega$, the claim follows from Theorem 7.1. $\square$

Corollary 7.5. *Let $\Omega \subseteq \mathbb{C}$ be open, and $f \in H(\Omega)$. Suppose that $z_0 \in \Omega$, $r > 0$, and $\overline{B}(z_0, r) \subseteq \Omega$, and define $\gamma(t) = z_0 + re^{it}$ for $t \in [0, 2\pi]$. Then*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\, dz \tag{11}$$

for all $n \in \mathbb{N}_0$. Moreover, if $|f(z)| \leq M$ for all $z \in \partial B(z_0, r)$, then for $n \in \mathbb{N}_0$ we have

$$|f^{(n)}(z_0)| \leq \frac{n!M}{r^n} \quad \text{(Cauchy estimates).}$$

Proof. Equation (11) follows from formula (10) in the proof of Theorem 7.1, since $a_n = \dfrac{f^{(n)}(z_0)}{n!}$ is the n-th Taylor coefficient of f at z_0.

Under the given assumptions, the Cauchy estimates follow from this:

$$\begin{aligned} |f^{(n)}(z_0)| &= \left| \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\, dz \right| \\ &\leq \frac{n!}{2\pi} \ell(\gamma) \frac{M}{r^{n+1}} = \frac{n!}{2\pi} 2\pi r \frac{M}{r^{n+1}} = \frac{n!M}{r^n}. \end{aligned}$$

$\square$

Definition 7.6. A function $f\colon \mathbb{C} \to \mathbb{C}$ that is holomorphic on the whole complex plane $\mathbb{C}$ is called an *entire function*.

Corollary 7.7. *Let f be an entire function. Then at each point z_0 we can represent f by a power series*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

that converges for all $z \in \mathbb{C}$.

Proof. This follows from Corollary 7.4. □

Theorem 7.8 (Liouville's Theorem)**.** *Every bounded entire function is constant.*

Proof. Let $f \in H(\mathbb{C})$ and suppose that there exists $M \geq 0$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Then f has a power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

that converges for all $z \in \mathbb{C}$. We know that

$$a_n = \frac{f^{(n)}(0)}{n!}$$

for all $n \in \mathbb{N}_0$. The Cauchy estimates with $z_0 = 0$ are valid for all $n \in \mathbb{N}_0$ and all $r > 0$. Hence

$$|a_n| = \frac{|f^{(n)}(0)|}{n!} \leq \frac{M}{r^n}$$

for all $n \in \mathbb{N}$ and all $r > 0$. Letting $r \to \infty$ (for fixed $n \in \mathbb{N}$), we see that $a_n = 0$ for $n \in \mathbb{N}$. So

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = a_0$$

for all $z \in \mathbb{C}$. This shows that f is a constant function. □

Theorem 7.9 (Fundamental Theorem of Algebra)**.** *Every non-constant complex polynomial has a root; more precisely, if $n \in \mathbb{N}$, $a_0, a_1, \ldots, a_n \in \mathbb{C}$, $a_n \neq 0$, and $P\colon \mathbb{C} \to \mathbb{C}$,*

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n \tag{12}$$

for $z \in \mathbb{C}$, then there exists $z_0 \in \mathbb{C}$ such that $P(z_0) = 0$.

Proof. Let P be a polynomial as in (12). We argue by contradiction and assume that $P(z) \neq 0$ for all $z \in \mathbb{C}$. Then $f = 1/P$ is an entire function.

Claim: f is bounded.

Obviously, $|P(z)| \to \infty$ as $|z| \to \infty$. Hence there exists $R > 0$ such that

$$|P(z)| > |P(0)| \quad \text{whenever } z \in \mathbb{C},\ |z| \geq R,$$

and so

$$|f(z)| < |f(0)| \quad \text{whenever } z \in \mathbb{C},\ |z| \geq R. \tag{13}$$

Since f is continuous and $\overline{B}(0, R)$ is compact, there exists $K \geq 0$ such that

$$|f(z)| \leq K \quad \text{for all } z \in \overline{B}(0, R).$$

So

$$|f(z)| \leq \max\{K, |f(0)|\} \quad \text{for all } z \in \mathbb{C}.$$

The claim follows.

Liouville's Theorem now implies that f is constant. This is a contradiction, since $|f(R)| < |f(0)|$ by (13). $\square$

Corollary 7.10. *The field $\mathbb{C}$ is algebraically closed.*

Remark 7.11. If $P \neq 0$ is a polynomial as in (12), where $a_n \neq 0$, then $n \in \mathbb{N}_0$ is called the *degree* of P, denoted by $\deg(P)$. For the constant polynomial $P = 0$ the degree is undefined.

If P and Q are polynomials and $Q \neq 0$, then we can divide P by Q with remainder by the usual division algorithm. Hence there exist polynomials R and S such that

$$P = S \cdot Q + R,$$

where $\deg(R) < \deg(Q)$ or $R = 0$.

If z_0 is a root of $P \neq 0$, then we can applies this to $Q(z) := z - z_0$, and we obtain

$$P(z) = S(z)(z - z_0) + R(z) \quad \text{for } z \in \mathbb{C},$$

where $\deg(R) < \deg(Q) = 1$ or $R = 0$. So R must be a constant polynomial, and $0 = P(z_0) = R(z_0)$ shows that $R = 0$. We conclude that

$$P(z) = S(z)(z - z_0) \quad \text{for } z \in \mathbb{C}.$$

If z_0 is also a root of S, we can repeat this reasoning, etc. It follows that there exists $k \in \mathbb{N}$ and a polynomial T with $T(z_0) \neq 0$ such that

$$P(z) = (z - z_0)^k T(z) \quad \text{for } z \in \mathbb{C}.$$

The number k is uniquely determined and called the *multiplicity* of the root z_0 of P.

Corollary 7.12. *Every polynomial P of degree $n \in \mathbb{N}$ has precisely n roots counting multiplicities; more precisely, if $z_1, \dots, z_l \in \mathbb{C}$, $l \in \mathbb{N}$, are the distinct roots of P and $k_1, \dots, k_l \in \mathbb{N}$ their respective multiplicities, then $k_1 + \cdots + k_l = n$.*

Moreover, there exists $a \in \mathbb{C}$, $a \neq 0$, such that

$$P(z) = a(z - z_1)^{k_1} \dots (z - z_m)^{k_m} \quad \text{for } z \in \mathbb{C}.$$

Proof (Outline). This follows from induction on n. For the induction step let P be a polynomial of degree n. Then P has a root z_0 by the Fundamental Theorem of Algebra. Then $P(z) = (z - z_0)Q(z)$ for $z \in \mathbb{C}$, where $\deg(Q) = n - 1$. Now we apply the induction hypothesis to Q. □

Theorem 7.13 (Morera's Theorem)**.** *Let $\Omega \subseteq \mathbb{C}$ be open, and $f \colon \Omega \to \mathbb{C}$ be continuous. Suppose that for every closed oriented triangle $\Delta \subseteq \Omega$ we have*

$$\int_{\partial\Delta} f = 0.$$

Then f is holomorphic on Ω.

This can be considered as a "converse" of Goursat's lemma.

Proof. It is enough to show that f is holomorphic on every open disk $B \subseteq \Omega$. If B is such a disk, fix $a \in B$ and define

$$F(w) = \int_{[a,w]} f(z)\, dz \quad \text{for } w \in B.$$

This makes sense, since B is convex. As in the proof of Corollary 6.4, one shows that $F \in H(B)$ and $F'(w) = f(w)$ for $w \in B$. So in B the function f is the derivative of a holomorphic function. Hence f is holomorphic itself (Corollary 7.2). □

8 Zeros of holomorphic functions

Definition 8.1. Let (X,d) be a metric space, and $A \subseteq X$.

(a) A point $x \in X$ is called an *isolated point* of A if $x \in A$ and if there exists $\epsilon > 0$ such that $B(x,\epsilon) \cap A = \{x\}$.

(b) A point $x \in X$ is called a *limit point* of A if for all $\epsilon > 0$ the ball $B(x,\epsilon)$ contains infinitely many distinct points of A.

Remark 8.2. (a) A point $x \in X$ is a limit point of A if and only if there exists a sequence $\{x_n\}$ of distinct elements in A such that $x = \lim_{n\to\infty} x_n$.

(b) Every point $x \in A$ is either an isolated point of A or a limit point of A.

(c) A set $K \subseteq X$ is compact if and only if every infinite set $A \subseteq K$ has a limit point in K.

Theorem 8.3. *Let $\Omega \subseteq \mathbb{C}$ be a region, and $f \in H(\Omega)$. Define*

$$Z(f) := \{a \in \Omega : f(a) = 0\} \qquad \text{(the zero-set of } f\text{)}.$$

Then either $Z(f) = \Omega$ (iff $f \equiv 0$), or $Z(f)$ is countable, consists of isolated points, has no limit points in Ω.

In the latter case, for each $a \in Z(f)$ there exists $m \in \mathbb{N}$ (the multiplicity *or* order *of the zero a), and $g \in H(\Omega)$ with $g(a) \neq 0$ such that*

$$f(z) = (z-a)^m g(z) \quad \textit{for } z \in \Omega. \tag{14}$$

Proof. Let $a \in Z(f)$, and fix $r > 0$ such that $B(a,r) \subseteq \Omega$. By Corollary 7.4 we can represent f by a power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

for $z \in B(a,r)$. Here $a_0 = f(a) = 0$. Now either $a_n = 0$ for all $n \in \mathbb{N}$, or there exists a smallest $m \in \mathbb{N}$ such that $a_m \neq 0$.

In the first case, $B(a,r) \subseteq Z(f)$. In the second case, we define

$$g(z) := \begin{cases} \dfrac{1}{(z-a)^m} f(z) & \text{for } z \in \Omega \setminus \{a\}, \\[2ex] \displaystyle\sum_{n=0}^{\infty} a_{n+m}(z-a)^n & \text{for } z \in B(a,r). \end{cases}$$

The definitions for g agree in

$$(\Omega \setminus \{a\}) \cap B(a,r) = B(a,r) \setminus \{a\},$$

and show that $g \in H(\Omega)$. Moreover, $g(a) = a_m \neq 0$ and

$$f(z) = (z-a)^m g(z) \quad \text{for } z \in \Omega. \tag{15}$$

By continuity of g at a, there exists $\epsilon > 0$ such that $g(z) \neq 0$ for $z \in B(a,\epsilon) \subseteq \Omega$. Then (15) shows that $f(z) \neq 0$ for $z \in B(a,\epsilon) \setminus \{a\}$, and so a is an isolated point of $Z(f)$.

To summarize, either a whole neighborhood of $a \in Z(f)$ belongs to $Z(f)$ or a is an isolated point of $Z(f)$.

Now let A be the set of limit points of $Z(f)$ in Ω. We want to show that A is open and relatively closed in Ω.

If $a \in A$, then there exists a sequence $\{z_n\}$ of distinct points in $Z(f)$ such that $a = \lim_{n\to\infty} z_n$ (Remark 8.2 (a)). By continuity of f we then have

$$f(a) = \lim_{n\to\infty} f(z_n) = 0,$$

and so $a \in Z(f)$. Then a is not an isolated point of $Z(f)$, and so there exists $r > 0$ such that $B(a,r) \subseteq Z(f)$ by the first part of the proof. Then $B(a,r) \subseteq A$ which shows that A is an open set.

To establish that A is relatively closed in Ω we verify that $\Omega \setminus A$ is open. To see this, let $b \in \Omega \setminus A$ be arbitrary. Then b is not a limit point of $Z(f)$, and so by the first part of the proof there exists $\epsilon > 0$ such that $B(b,\epsilon) \subseteq \Omega$, and $B(b,\epsilon)$ contains at most one point in $Z(f)$. Then $B(b,\epsilon) \subseteq \Omega \setminus A$.

We have shown that A is open and relatively closed in Ω. Since Ω is a region, we conclude that $A = \Omega$ or $A = \emptyset$. In the first case, $\Omega = A \subseteq Z(f) \subseteq \Omega$, and so $Z(f) = \Omega$.

In the second case, $A = \emptyset$ and so $Z(f)$ has no limit points in Ω. By the first part of the proof, we also see that $Z(f)$ consists of isolated points, and that at each point $a \in Z(f)$ the function f has a representation as in (15).

It remains to show that if $Z(f) \neq \Omega$, then $Z(f)$ is countable. The set Ω (as any open set in $\mathbb{C}$) has a *compact exhaustion*; namely, there are compact sets $K_n \subseteq \Omega$ for $n \in \mathbb{N}$ such that $K_n \subseteq K_{n+1}$ for $n \in \mathbb{N}$, and $\bigcup_{n\in\mathbb{N}} K_n = \Omega$. For example, we can take

$$K_n = \{z \in \Omega : |z| \leq n \text{ and } \operatorname{dist}(z, \partial\Omega) \geq 1/n\}.$$

Now $Z(f) \cap K_n$ must be a finite set for each $n \in \mathbb{N}$, for otherwise $Z(f)$ would have a limit point in K_n (see Remark 8.2 (c)), and hence in Ω. So

$$Z(f) = \bigcup_{n \in \mathbb{N}} (K_n \cap Z(f))$$

is a countable union of finite sets, which implies that $Z(f)$ is countable. $\square$

Example 8.4. (a) In Theorem 8.3 it is important that Ω is a region. For example, let $\Omega = B(0,1) \cup B(1,1)$ and

$$f(z) := \begin{cases} 0 & \text{for } z \in B(0,1), \\ \\ e^z & \text{for } z \in B(1,1), \end{cases}$$

Then f is holomorphic on Ω. We have $Z(f) = B(0,1) \neq \Omega$, but every point in $B(0,1)$ is a limit point of $Z(f)$ in Ω.

(b) Let $f(z) = \exp(1/z) - 1$ for $z \in \Omega := \mathbb{C} \setminus \{0\}$. Then $f \in H(\Omega)$, and $Z(f) := \{1/(2\pi i n) : n \in \mathbb{Z} \setminus \{0\}\}$. Then $Z(f) \neq \Omega$ has a limit point, namely 0, but this point does not lie in Ω.

Theorem 8.5 (Uniqueness Theorem)**.** *Let $\Omega \subseteq \mathbb{C}$ be a region, and $f, g \in H(\Omega)$. If the set*

$$\{z \in \Omega : f(z) = g(z)\}$$

has a limit point in Ω, then $f = g$.

Proof. Consider $h = f - g \in H(\Omega)$. Then the set

$$\begin{aligned} Z(h) &= \{z \in \Omega : h(z) = 0\} \\ &= \{z \in \Omega : f(z) = g(z)\} \end{aligned}$$

has a limit point in Ω. So $Z(h) = \Omega$ by Theorem 8.3 which implies $h = f - g = 0$. Hence $f = g$. $\square$

Theorem 8.6 (Maximum Modulus Theorem. Version I)**.** *Let $\Omega \subseteq \mathbb{C}$ be a region, and $f \in H(\Omega)$. If $|f|$ attains a local maximum at a point in Ω, then f is a constant function.*

Proof. Suppose f attains a local maximum at $a \in \Omega$. Then there exists $\epsilon > 0$ such that $B(a, \epsilon) \subseteq \Omega$ and

$$|f(a)| \geq |f(z)|$$

for all $z \in B(a, \epsilon)$. Replacing f by $e^{i\theta} f$ with suitable $\theta \in [0, 2\pi]$, we may assume $f(a) \geq 0$ without loss of generality. Then

$$f(a) = |f(a)| \geq |f(z)| \geq |\operatorname{Re}(f(z))|$$

for $z \in B(a, \epsilon)$.

Pick $r \in (0, \epsilon)$ and let $\gamma(t) = a + re^{it}$, $t \in [0, 2\pi]$. Then by Cauchy's Integral Formula we have

$$\begin{aligned} f(a) &= \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a + re^{it})}{re^{it}} ire^{it}\, dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{it})\, dt \quad \text{(Mean Value Property)} \end{aligned}$$

(the average of the function values on the circle γ is equal to the function value at the center of the circle!). Hence

$$\begin{aligned} f(a) = \operatorname{Re}(f(a)) &= \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re}\left(f(a + re^{it})\right) dt \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \left|f(a + re^{it})\right| dt \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} |f(a)|\, dt = |f(a)| = f(a). \end{aligned}$$

So we must have equality in all these inequalities, which shows that

$$\begin{aligned} \int_0^{2\pi} \left(\left|f(a + re^{it})\right| - \operatorname{Re}\left(f(a + re^{it})\right)\right) dt = 0, \\ \int_0^{2\pi} \left(|f(a)| - \left|f(a + re^{it})\right|\right) dt = 0. \end{aligned}$$

Since the integrands in these integrals are non-negative continuous functions, we conclude that

$$\operatorname{Re}\left(f(a + re^{it})\right) = \left|f(a + re^{it})\right| = |f(a)| = f(a)$$

for all $t \in [0, 2\pi]$. In particular,

$$\operatorname{Im}\left(f(a + re^{it})\right) = 0,$$

and so

$$f(a+re^{it}) = \operatorname{Re}\big(f(a+re^{it})\big) = f(a)$$

for all $t \in [0, 2\pi]$. This shows that the function f takes the constant value $f(a)$ on the circle $\partial B(a,r)$. Hence $f \equiv f(a)$ as follows from the Uniqueness Theorem, and so f is a constant function. $\square$

We record two important facts that were derived in the proof of the last theorem.

Corollary 8.7. *Let $\Omega \subseteq \mathbb{C}$ be a region, $f \in H(\Omega)$, and $a \in \Omega$ and $r > 0$ be such that $\overline{B}(a,r) \subseteq \Omega$. Then we have:*

(a) $f(a) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(a+re^{it})\,dt$ (Mean Value Property),

(b) $|f(a)| \leq \max\{|f(a+re^{it})| : t \in [0,2\pi]\}$ *with equality if and only if f is a constant function.*

Proof. In the proof of Theorem 8.6 statement (a) was proved explicitly, and statement (b) easily follows from the considerations in this proof. $\square$

Corollary 8.8. *Let $\Omega \subseteq \mathbb{C}$ be a region, $f \in H(\Omega)$, and $a \in \Omega$ and $r > 0$ be such that $\overline{B}(a,r) \subseteq \Omega$. If $f(z) \neq 0$ for all $z \in B(a,r)$, then we have*

$$|f(a)| \geq \min\{|f(a+re^{it})| : t \in [0,2\pi]\} \tag{16}$$

with equality if and only if f is a constant function.

Proof. If f has a zero on $\partial B(a,r)$, then the inequality is obvious with strict inequality.

In the other case, where f has no zero on $\partial B(a,r)$, we have $f(z) \neq 0$ for all $z \in \overline{B}(a,r)$ by our hypotheses. Hence there exists $R > r$ such that $B(a,R) \subseteq \Omega$ and f is zero-free on $\widetilde{\Omega} := B(a,R)$. For otherwise, we could find a sequence $\{z_n\}$ of zeros of f with $|z_n - a| \to r$. Passing to a subsequence if necessary, we may without loss of generality assume that $\{z_n\}$ converges, say $z_n \to w$. Then

$$|w-a| = \lim_{n\to\infty} |z_n - a| = r,$$

and so $w \in \partial B(a,r) \subseteq \Omega$. Moreover, the continuity of f shows that

$$f(w) = \lim_{n\to\infty} f(z_n) = 0.$$

This contradicts the fact that f has no zero on $\partial B(a, r)$.

The function $g := 1/f$ is then holomorphic on $\widetilde{\Omega}$, and so by Corollary 8.7 (b) we have

$$|g(a)| \leq \max\{|g(a + re^{it})| : t \in [0, 2\pi]\}. \tag{17}$$

By taking reciprocals here, we get (16). If we have equality in (16), then we must have equality in (17). This implies that g is constant on $\widetilde{\Omega}$. So $f = 1/g$ is constant on $\widetilde{\Omega}$, and hence constant on Ω by the Uniqueness Theorem. $\square$

Theorem 8.9 (Maximum Modulus Theorem. Version II)**.** *Let $\Omega \subseteq \mathbb{C}$ be a bounded region, and $f : \overline{\Omega} \to \mathbb{C}$ be a continuous function that is holomorphic on Ω. Then*

$$\max\{|f(z)| : z \in \overline{\Omega}\} = \max\{|f(z)| : z \in \partial\Omega\} \tag{18}$$

The theorem says that the maximum of $|f|$ on $\overline{\Omega}$ (which exists by continuity of $|f|$ and compactness of $\overline{\Omega}$) is attained at some point on the boundary $\partial\Omega$ of Ω. Note that $\partial\Omega \neq \emptyset$, because the only open sets $U \subseteq \mathbb{C}$ with $\partial U = \emptyset$ are $U = \mathbb{C}$ and $U = \emptyset$ (exercise!).

Proof. The maximum on the left-hand side of (18) is attained at some point $z_0 \in \overline{\Omega}$. If $z_0 \in \partial\Omega$, then (18) holds.

If $z_0 \in \Omega$, then f is constant on Ω by Theorem 8.6. Then f is also constant on $\overline{\Omega}$ by the continuity of f (exercise!). Since $\partial\Omega \neq \emptyset$, again (18) holds. $\square$

9 The Open Mapping Theorem

Theorem 9.1. *Let $\Omega \subseteq \mathbb{C}$ be open, $\varphi \in H(\Omega)$, and $z_0 \in \Omega$. If $\varphi'(z_0) \neq 0$, then φ is locally injective near z_0 with a holomorphic inverse map.*

More precisely, there exists an open neighborhood $V \subseteq \Omega$ of z_0 such that

(i) *φ is injective on V and $\varphi'(z) \neq 0$ for $z \in V$,*

(ii) *$W = \varphi(V)$ is an open set,*

(iii) *the inverse map $\psi := (\varphi|V)^{-1}$ is holomorphic on W.*

Proof. Since $\varphi \in H(\Omega)$, we have $\varphi' \in H(\Omega)$ by Corollary 7.2. In particular, φ' is continuous at z_0. Hence there exists $\delta > 0$ such that $\overline{B}(z_0, \delta) \subseteq \Omega$ and

$$|\varphi'(z) - \varphi'(z_0)| \leq \frac{1}{2}|\varphi'(z_0)| \quad \text{for all } z \in \overline{B}(z_0, \delta). \tag{19}$$

Define $V := B(z_0, \delta)$. Then V is an open convex neighborhood of z_0.

For $z_1, z_2 \in V$ we have $[z_1, z_2] \subseteq V$, and so

$$\begin{aligned}
|\varphi(z_1) - \varphi(z_2)| &= \left| \int_{[z_1,z_2]} \varphi'(z)\, dz \right| \\
&= \left| \int_{[z_1,z_2]} \varphi'(z_0)\, dz + \int_{[z_1,z_2]} (\varphi'(z) - \varphi'(z_0))\, dz \right| \\
&\geq \left| \int_{[z_1,z_2]} \varphi'(z_0)\, dz \right| - \left| \int_{[z_1,z_2]} (\varphi'(z) - \varphi'(z_0))\, dz \right| \qquad (20) \\
&\geq |z_1 - z_2| \cdot |\varphi'(z_0)| - |z_1 - z_2| \cdot \max_{z \in [z_1,z_2]} |\varphi'(z) - \varphi(z_0)| \\
&\geq \frac{1}{2}|\varphi'(z_0)| \cdot |z_1 - z_2| \quad \text{by (19)}.
\end{aligned}$$

This inequality shows that φ is injective on V.

Moreover, inequality (19) implies that

$$\begin{aligned}
|\varphi'(z)| &= |\varphi'(z_0) + (\varphi'(z) - \varphi'(z_0))| \\
&\geq |\varphi'(z_0)| - |\varphi'(z) - \varphi'(z_0)| \\
&\geq \frac{1}{2}|\varphi'(z_0)|
\end{aligned}$$

for $z \in V$. Hence $\varphi'(z) \neq 0$ for all $z \in V$. We have established (i).

In order to prove (ii), let $w \in W = \varphi(V)$ be arbitrary. Then there exists $a \in V$ such that $w = \varphi(a)$. Pick $r > 0$ such that $\overline{B}(a, r) \subseteq V$. Then by (20) we have

$$|\varphi(a + re^{it}) - \varphi(a)| \geq \frac{1}{2}|\varphi'(z_0)|r \quad \text{for } t \in [0, 2\pi]. \tag{21}$$

Let $\epsilon := \frac{1}{6}|\varphi'(z_0)|r > 0$. We claim that $B(w, \epsilon) \subseteq W$. Indeed, if $u \in B(w, \epsilon)$ is arbitrary, then

$$\begin{aligned}|\varphi(a) - u| = |w - u| < \epsilon \leq 2\epsilon - |\varphi(a) - u|& \\ < \min_{t\in[0,2\pi]} |\varphi(a + re^{it}) - \varphi(a)| - |\varphi(a) - u| \quad &\text{(by (21))} \\ \leq \min_{t\in[0,2\pi]} |\varphi(a + re^{it}) - u|.&\end{aligned}$$

By Corollary 8.8 this is only possible if $\varphi - u$ has a zero in $B(a, r)$. Hence there exists $z \in B(a, r) \subseteq V$ such that $u = \varphi(z)$. This implies that $u \in \varphi(V)$, and so $B(a, \epsilon) \subseteq W$. It follows that W is open.

To prove (iii) note that (20) can be rewritten as

$$|\psi(w_1) - \psi(w_2)| \leq \frac{2}{|\varphi'(z_0)|} \cdot |w_1 - w_2| \quad \text{for } w_1, w_2 \in W.$$

This shows that $\psi\colon W \to V$ is continuous on W.

In order to prove that $\psi \in H(W)$, let $w \in W$ be arbitrary. Then there exists a unique point $z \in V$ with $\varphi(z) = w$. We claim that

$$\psi'(w) = \frac{1}{\varphi'(z)}. \tag{22}$$

Note that $\varphi'(z) \neq 0$ by (i). To prove (22), we have to show that if $\{w_n\}$ is an arbitrary sequence in $W \setminus \{w\}$ with $w_n \to w$, then

$$\lim_{n\to\infty} \frac{\psi(w_n) - \psi(w)}{w_n - w} = \frac{1}{\varphi'(z)}.$$

If $z_n := \psi(w_n)$ for $n \in \mathbb{N}$, then $z_n \to \psi(w) = z$ by the continuity of ψ. Therefore,

$$\lim_{n\to\infty} \frac{\psi(w_n) - \psi(w)}{w_n - w} = \lim_{n\to\infty} \frac{z_n - z}{\varphi(z_n) - \varphi(z)} = \frac{1}{\varphi'(z)}.$$

It follows that $\psi \in H(W)$. The proof is complete. $\square$

Example 9.2. Let $m \in \mathbb{N}$ and $\pi_m\colon \mathbb{C} \to \mathbb{C}$ be defined by $\pi_m(z) = z^m$ for $z \in \mathbb{C}$ (the m-th power function). Then every point $w = Re^{i\beta} \neq 0$, $R > 0$, $\beta \in \mathbb{R}$, has the m distinct preimages

$$z_k = R^{1/m} e^{i\beta/m + 2\pi ik/m}, \quad k \in \{0, \dots, m-1\}.$$

This is also true for the point 0 if we count its only preimage 0 with multiplicity m. So every point has exactly m preimages counting multiplicities and π_m is "m-to-1".

As the next theorem shows, a similar statement is true locally for every non-constant holomorphic function.

Theorem 9.3. *Let $\Omega \subseteq \mathbb{C}$ be open, $f \in H(\Omega)$ be non-constant near $z_0 \in \Omega$, and $w_0 = f(z_0)$. Let $m \in \mathbb{N}$ be the order of the zero of the function $g := f - w_0$ at z_0. Then there exists an open neighborhood $V \subseteq \Omega$ of z_0, and $\varphi \in H(V)$ such that*

(i) $g(z) = f(z) - w_0 = \varphi(z)^m$ *for* $z \in V$,

(ii) φ *is a bijective map from V onto a disk $B(0, r)$ with $r > 0$, $\varphi(z_0) = 0$, and $\varphi'(z) \neq 0$ for $z \in V$.*

If we define $U = B(w_0, r^m)$, $\psi(w) = w - w_0$, and let π_m be the m-th power function as in Example 9.2, then we get a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\quad f \quad} & U \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ B(0,r) & \xrightarrow{\pi_m} & B(0,r^m). \end{array}$$

Here the maps φ and ψ can be considered as bijective holomorphic "changes of coordinates". So near z_0 the map f behaves in the same way as π_m and is locally m-to-1.

Proof. Without loss of generality, Ω is a convex open neighborhood of z_0 such that $f(z) - w_0 \neq 0$ for $z \in \Omega \setminus \{z_0\}$ (this is based on Theorem 8.3). Then

$$f(z) - w_0 = (z - z_0)^m \cdot h(z) \quad \text{for } z \in \Omega,$$

where $h \in H(\Omega)$ and $h(z) \neq 0$ for all $z \in \Omega$ (see Theorem 8.3).

We have $h'/h \in H(\Omega)$, and so h'/h has a primitive $k \in H(\Omega)$ by Corollary 6.4 which means $k' = h'/h$.

Then
$$(he^{-k})' = h'e^{-k} - hk'e^{-k} = h'e^{-k} - h'e^{-k} = 0,$$
and so he^{-k} is a constant function. Then $he^{-k} \equiv h(z_0)e^{-k(z_0)} \neq 0$. There exists $c \in \mathbb{C}$ such that $e^c = h(z_0)e^{-k(z_0)} = he^{-k}$. Define $l := k + c$. Then $l \in H(\Omega)$ and
$$e^l = e^k e^c = e^k he^{-k} = h.$$
Define $\varphi(z) := (z-z_0)e^{l(z)/m}$ for $z \in \Omega$. Then $\varphi \in H(\Omega)$ and $g = f - w_0 = \varphi^m$.

Moreover, $\varphi(z_0) = 0$ and $\varphi'(z_0) = e^{l(z_0)/m} \neq 0$. By Theorem 9.1 we can choose an open neighborhood $\widetilde{V}$ of z_0 such that $\varphi|\widetilde{V}$ is a bijection onto some open neighborhood W of $\varphi(z_0) = 0$ and such that $\varphi'(z) \neq 0$ for $z \in \widetilde{V}$. Now pick $r > 0$ such that $B(0, r) \subseteq W$ and let $V := \varphi^{-1}(B(0, r))$. Then V and φ (restricted to V) have the desired properties. □

Remark 9.4. The number m in Theorem 9.3 is called the *local degree* of f at z_0, written $m = \deg_f(z_0)$. Note that

$$f'(z_0) = m\varphi'(z_0) \cdot \varphi(z_0)^{m-1} \begin{cases} \neq 0 & \text{for } m = 1, \\ \\ = 0 & \text{for } m \geq 2. \end{cases}$$

So $m = \deg_f(z_0) \geq 2$ if and only if $f'(z_0) = 0$.

The zeros of f' are called the *critical points* of f. Each image point of a critical value is called a *critical value* of f. The critical points of f are precisely the points where the local degree is greater than 1. If f is a non-constant holomorphic function on a region Ω, then the set of critical points of f consists of isolated points and has no limit point in Ω as follows from the Uniqueness Theorem.

Corollary 9.5 (Open Mapping Theorem)**.** *Let $\Omega \subseteq \mathbb{C}$ be a region, and $f \in H(\Omega)$ be non-constant. Then f is an open map, i.e., the image $f(U)$ of every open set $U \subseteq \Omega$ is open. Moreover, the image $f(U)$ of every region $U \subseteq \Omega$ is a region.*

Proof. The first part follows from Theorem 9.3. So if $U \subseteq \Omega$ is a region, then $f(U)$ is open. Moreover, $f(U)$ is also connected as the continuous image of a connected set. Hence $f(U)$ is a region. □

Corollary 9.6. *Let $\Omega \subseteq \mathbb{C}$ be a region, and $f\colon \Omega \to \mathbb{C}$ be a holomorphic function that is a bijection onto its image $\widetilde{\Omega} := f(\Omega)$ (i.e., $f\colon \Omega \to \widetilde{\Omega}$ is a bijection). Then $f'(z) \neq 0$ for all $z \in \Omega$, the inverse map $g := f^{-1}\colon \widetilde{\Omega} \to \Omega$ is holomorphic on the open set $\widetilde{\Omega}$, and we have*

$$g'(w) = \frac{1}{f'(g(w))} \quad \text{for all } w \in \widetilde{\Omega}. \tag{23}$$

Proof. Since f is a bijection, the local degree of f must be equal to 1 at each point of Ω. Hence $f'(z) \neq 0$ for all $z \in \Omega$ by Remark 9.4. By Corollary 9.5 the set $\widetilde{\Omega}$ is open, and $g = f^{-1}$ is holomorphic on $\widetilde{\Omega}$ by Theorem 9.1 (iii).

Since $f(g(w)) = w$ for all $w \in \widetilde{\Omega}$, we conclude from the chain rule that

$$f'(g(w)) \cdot g'(w) = 1$$

for $w \in \widetilde{\Omega}$. So (23) follows. $\square$

10 Elementary functions

10.1. Trigonometric functions. How can we define $\sin z$ and $\cos z$ for $z \in \mathbb{C}$ so that

(i) the definitions agree with the usual definition for $z \in \mathbb{R}$,

(ii) sin and cos are holomorphic on $\mathbb{C}$?

By the Uniqueness Theorem this is possible in at most one way, and by Corollary 7.7 the functions sin and cos should have power series representations converging on $\mathbb{C}$.

So we define

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \quad \text{and} \quad \cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

for $z \in \mathbb{C}$. These series converge for all $z \in \mathbb{C}$ by the ratio test; so by Theorem 5.9 the functions sin and cos are holomorphic on $\mathbb{C}$. Moreover, with these definitions we get the usual functions for $z \in \mathbb{R}$ by the well-known power series representations for sin and cos on $\mathbb{R}$.

All the standard trigonometric identities are also valid for complex arguments. For example, we have

$$\sin^2 z + \cos^2 z = 1 \quad \text{for } z \in \mathbb{C}.$$

Indeed, the functions on both sides of this equation are holomorphic functions of z on $\mathbb{C}$. They are the same for $z \in \mathbb{R}$. So by the Uniqueness Theorem they are the same for *all* $z \in \mathbb{C}$.

Similarly, we have

$$\sin(z+w) = \sin z \cos w + \cos z \sin w \tag{24}$$

for $z, w \in \mathbb{C}$.

Indeed, we know that (24) holds for $z, w \in \mathbb{R}$. Fix $w \in \mathbb{R}$. Then

$$z \mapsto \sin(z+w) \quad \text{and} \quad z \mapsto \sin z \cos w + \cos z \sin w$$

are holomorphic functions on $\mathbb{C}$ that are identical on $\mathbb{R}$. So they have the same values for all $z \in \mathbb{C}$. We conclude that (24) holds for all $z \in \mathbb{C}$, $w \in \mathbb{R}$.

Now fix $z \in \mathbb{C}$ and consider the right-hand and the left-hand side of (24) as holomorphc functions of w on $\mathbb{C}$. They agree for $w \in \mathbb{R}$, and so they agree for all $w \in \mathbb{C}$ by the uniqueness theorem. So (24) holds for all $z, w \in \mathbb{C}$.

We have

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \text{for } z \in \mathbb{C}. \tag{25}$$

Again we note that both sides in these equations are holomorphic functions of z on $\mathbb{C}$, and that the function agree for $z \in \mathbb{R}$. For example, for $z = t \in \mathbb{R}$ we have

$$\frac{e^{it} - e^{-it}}{2i} = \frac{(\cos t + i \sin t) - (\cos t - i \sin t)}{2i} = \sin t.$$

So the equations (25) are valid for all $z \in \mathbb{C}$.

An immediate consequence of (25) is the identity

$$e^{iz} = \cos z + i \sin z \quad \text{for } z \in \mathbb{C} \qquad \text{(Euler's formula)}.$$

What are the zeros of the functions sin and cos? Note that

$$\begin{aligned} \sin z = 0 \quad &\Leftrightarrow \quad e^{iz} - e^{-iz} = 0 \\ &\Leftrightarrow \quad e^{2iz} = 1 \\ &\Leftrightarrow \quad 2iz = 2\pi i k \text{ for some } k \in \mathbb{Z} \\ &\Leftrightarrow \quad z = \pi k \text{ for some } k \in \mathbb{Z}. \end{aligned}$$

Since $\cos z = \sin(\frac{\pi}{2} - z)$ for $z \in \mathbb{C}$, we have

$$\cos z = 0 \Leftrightarrow z = \frac{\pi}{2} + k\pi \text{ for some } k \in \mathbb{Z}.$$

We can compute $\sin z$ and $\cos z$ for $z = x + iy$, $x, y \in \mathbb{R}$, by using the addition formulas. First note that

$$\cos(iy) = \frac{e^{-y} + e^{y}}{2} = \cosh y, \quad \sin(iy) = \frac{e^{-y} - e^{y}}{2i} = i \sinh y.$$

Hence

$$\begin{aligned} \cos(x + iy) &= \cos x \cos(iy) - \sin x \sin(iy) \\ &= \cos x \cosh y - i \sin x \sinh y, \end{aligned}$$

and similarly

$$\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y.$$

The functions tan and cot are defined in the usual way. So

$$\tan z := \frac{\sin z}{\cos z}$$

is a holomorphic function of z on $\mathbb{C} \setminus \{k\pi + \pi/2 : k \in \mathbb{Z}\}$ and

$$\cot z := \frac{\cos z}{\sin z}$$

is a holomorphic function of z on $\mathbb{C} \setminus \{k\pi : k \in \mathbb{Z}\}$.

10.2. Logarithms. Every solution of the equation $z = e^w$ is called a *logarithm* of z. A number $z \in \mathbb{C}$ has a logarithm if and only if $z \neq 0$. Let $z = re^{i\alpha}$, $r > 0$, $\alpha \in \mathbb{R}$. Then

$$z = e^{\log r + i\alpha} = e^w$$

iff $w = \log r + i\alpha + 2\pi i k$, $k \in \mathbb{Z}$. So every $z \in \mathbb{C} \setminus \{0\}$ has infinitely many logarithms.

A continuous function $L\colon \Omega \to \mathbb{C}$ defined on a region $\Omega \subseteq \mathbb{C}$ is called a *branch of the logarithm* if $e^{L(z)} = z$ for all $z \in \Omega$. Note that $0 \notin \Omega$ if a branch of the logarithm exists, but this condition is not sufficient for the existence of the branch of the logarithm on a region Ω. For example, if $\Omega = \mathbb{C} \setminus \{0\}$, then no branch of the logarithm exists on Ω.

If $L_1, L_2\colon \Omega \to \mathbb{C}$ are two branches of the logarithm, then $e^{L_1(z)} = e^{L_2(z)}$, and so $e^{L_1(z) - L_2(z)} = 1$ for all $z \in \Omega$. It follows that

$$(L_1 - L_2)(\Omega) \subseteq \{2\pi i k : k \in \mathbb{Z}\}.$$

Since $(L_1 - L_2)(\Omega)$ is a connected set, we conclude that there exists $k \in \mathbb{Z}$ such that

$$L_1(z) = L_2(z) + 2\pi i k$$

for all $z \in \Omega$. So the branches of the logarithm in a region are obtained from one branch (if it exists) by adding $2\pi i k$, $k \in \mathbb{Z}$. In particular, knowing the function value of a branch at one point determines the branch uniquely.

The exponential function maps the strip

$$S := \{w \in \mathbb{C} : -\pi < \mathrm{Im}(w) < \pi\}$$

bijectively onto the slit plane $\mathbb{C} \setminus (-\infty, 0]$. By Corollary 9.6 the function $\exp|S$ has a holomorphic inverse function denoted by log and called the *principal branch of the logarithm*. It is the unique branch of the logarithm on $\Omega = \mathbb{C} \setminus (-\infty, 0]$ with $\log 1 = 0$.

By Corollary 9.6 we have

$$\frac{d}{dz} \log z = \frac{1}{e^{\log z}} = \frac{1}{z}.$$

The expression $\log z$ agrees with the usual definition for $z > 0$. Note that log is discontinuous along the "branch cut" $(-\infty, 0]$. Indeed, if for fixed $r > 0$ the points $z_1 = re^{i\alpha_1}$ and $z_1 = re^{i\alpha_2}$ with $\alpha_1, \alpha_2 \in (-\pi, \pi)$ approach $-r$ from the upper and lower half-planes, respectively, then $\alpha_1 \to \pi$ and $\alpha_2 \to -\pi$. So

$$\log(z_1) = \log r + i\alpha_1 \to \log r + i\pi$$

and

$$\log(z_2) = \log r + i\alpha_2 \to \log r - i\pi.$$

This shows that the values of the principal branch of the logarithm differ by $2\pi i$ on "different sides" of the branch cut.

Note that in general we have

$$\log(z_1 z_2) \neq \log(z_1) + \log(z_2)$$

if log is the principal branch of the logarithm. For example, if $z_1 = z_2 = i \in \mathbb{C} \setminus (-\infty, 0]$, then $z_1 z_2 = i^2 = -1$ is not even contained in the domain of definition $\mathbb{C} \setminus (-\infty, 0]$ of the principal branch.

10.3. Roots and powers. If $\Omega \subseteq \mathbb{C}$ is a region where a branch L of the logarithm exists, we define for $a \in \mathbb{C}$ and $z \in \Omega$,

$$z^a := \exp(aL(z)).$$

Note that if $\widetilde{L}$ is another branch of the logarithm, then $\widetilde{L} = L + 2\pi ik$, $k \in \mathbb{Z}$, and so

$$\exp(a\widetilde{L}(z)) = \exp(aL(z) + a2\pi ik) = \exp(aL(z)) \exp(a2\pi ik)$$

for $z \in \Omega$. So different branches of the power function z^a differ by a multiplicative factor. If $a \in \mathbb{Z}$, then $\exp(a2\pi ik) = 1$, and so there exists only one branch. So for $a = n \in \mathbb{Z}$, $z \neq 0$,

$$z^a = \exp(aL(z)) = \big(\exp(L(z)\big)^a = z^n$$

agrees with the power function as defined in Section 1.

Unless otherwise stated, z^a will denote the *principal branch* defined on $\mathbb{C} \setminus (-\infty, 0]$ as

$$z^a := \exp(a \log(z)),$$

where log is the principal branch of the logarithm. This branch is normalized so that

$$1^a = \exp(a \log(0)) = \exp(0) = 1,$$

and agrees with the usual definition of the power function for $z > 0$, $a \in \mathbb{R}$.

Note that

$$\begin{aligned}\frac{d}{dz} z^a &= \frac{d}{dz} \exp(a \log(z)) = a \frac{1}{z} \exp(a \log(z)) \\ &= a \exp(-\log(z)) \exp(a \log(z)) = a \exp((a-1) \log(z)) = a z^{a-1}\end{aligned}$$

for $z \in \mathbb{C} \setminus (-\infty, 0]$.

If $z = re^{i\alpha}$, $r > 0$, $\alpha \in (-\pi, \pi)$, we have

$$z^a = \exp(a \log(re^{i\alpha})) = \exp(a(\log r + i\alpha)).$$

For $a > 0$ this means that

$$z^a = \exp(a(\log r + i\alpha)) = r^a e^{i\alpha a}.$$

In particular, a sector

$$S = \{re^{i\alpha} : R > 0,\ \varphi_1 < \alpha < \varphi_2\},$$

where $-\pi \leq \varphi_1 < \varphi_2 \leq \pi$ and whose opening angle $\varphi_2 - \varphi_1$ is sufficently small, is mapped by $z \mapsto z^a$ onto the sector

$$S' = \{\rho e^{i\beta} : \rho > 0,\ a\varphi_1 < \alpha < a\varphi_2\}$$

whose opening angle differs from the opening angle of S by the factor a.

If $a = 1/n$, where $n \in \mathbb{N}$, we also use the notation

$$\sqrt[n]{z} := z^{1/n} = z^a.$$

For the principal branch we get

$$\sqrt[n]{re^{i\alpha}} = r^{1/n} e^{i\alpha/n}$$

for $z = re^{i\alpha}$, $r > 0$, $\alpha \in (-\pi, \pi)$.

Note that if $r > 0$ is fixed and $z_1 = re^{i\alpha}$ approaches the point $-r$ on the branch cut $(-\infty, 0]$ from the upper half-plane, then $\alpha \to \pi$ and so $\sqrt[n]{z_1} = \sqrt[n]{re^{i\alpha}} \to r^{1/n}e^{i\pi n}$. Similarly, if $z_2 = re^{i\alpha}$ approaches the point $-r$ on the branch cut from the lower half-plane, then $\alpha \to -\pi$ and so $\sqrt[n]{z_2} = \sqrt[n]{re^{i\alpha}} \to r^{1/n}e^{i\pi n} \to r^{1/n}e^{-i\pi n}$.

In particular, if $n = 2$, then the values of $\sqrt{z}$ for $z = -r$ on "different sides of the branch cut" are $\sqrt{r}i$ and $-\sqrt{r}i$. So these values differ by a factor -1.

10.4. Riemann surfaces spread over the plane. How can we get rid of the artificial branch cut of the logarithm? We cannot define the logarithm as a "sensible" (for example, continuous) function on $\mathbb{C} \setminus \{0\}$. Indeed, each $z = re^{i\alpha} \in \mathbb{C} \setminus \{0\}$ as infinitely many logarithms

$$\log r + i\alpha + 2\pi i k, \quad k \in \mathbb{Z}, \tag{26}$$

and these values cannot be patched up to a continuous function on $\mathbb{C} \setminus \{0\}$. So the logarithm is a "multi-valued" function. To get a well-defined function in the usual sense, one introduces a surface with infinitely many layers (roughly speaking, one layer for each $k \in \mathbb{Z}$ in (26)) so that on this surface the logarithm is a well-defined and continuous function.

One way to do this is as follows. Let $S = \{(z, w) \in \mathbb{C}^2 : z = \exp(w)\}$. Then we may think of S as a "covering surface" of $\mathbb{C} \setminus \{0\}$ where each point $(z, w) \in S$ lies "above" the point z. The logarithm is then defined as the map $(z, w) \in S \mapsto w$.

Another way to visualize this *Riemann surface spread over* $\mathbb{C} \setminus \{0\}$ is by the following gluing procedure. For $k \in \mathbb{Z}$ let

$$P_k = \{re^{i\alpha} : r > 0,\ 2\pi i(k-1) \leq \alpha \leq 2\pi i(k+1)\}$$

be a copy of punctured complex plane, where the points in P_k with $\alpha = 2\pi i(k \pm 1)$ on different sides of the branch cut $(-\infty, 0]$ are considered as distinct. Then the Riemann surface S of the logarithm is obtained by gluing together (i.e., identifying) each point $re^{i2\pi i(k+1)}$, $r > 0$, on the upper side of the branch cut of P_k with the same point on lower side of the branch cut of P_{k+1}. On a more intuitive level, this surface has the shape of a "bi-infinite spiral staircase" lying above $\mathbb{C} \setminus \{0\}$.

The logarithm on this surface S is defined as follows. If $z \in S$, then $z \in re^{i\alpha} \in P_k$ for some $k \in \mathbb{Z}$, where $2\pi i(k-1) \leq \alpha \leq 2\pi i(k+1)$. Then we set $\log(z) := \log r + i\alpha$. This is a well-defined function on S.

10.5. The Riemann surface of $\sqrt{z}$. The square-root function has two branches on $\mathbb{C} \setminus (-\infty, 0]$ that differ by the factor ± 1. To get a well-defined square-root function we define a Riemann surface $S = \{(z, w) \in \mathbb{C}^2 : z = w^2\}$. Then by the projection map $(z, w) \in S \mapsto z$ this surface is spread over the complex plane $\mathbb{C}$. On S the square-root function is given by the map $(z, w) \in S \mapsto w$.

Similarly as for the logarithm, one can obtain S from a gluing procedure also as follows. We take two copies P_+ and P_- of the complex plane with a slit along the negative real axis $(-\infty, 0]$ as branch cut. In each copy we consider the points $re^{\pm i\pi}$, $r > 0$, on different sides of the branch cut as distinct. Then we glue P_+ and P_- together "cross-wise" so that the point $re^{+i\pi} \in P_+$ is identified with $re^{-i\pi} \in P_-$, and $re^{-i\pi} \in P_+$ is identified with $re^{+i\pi} \in P_-$. We also identify $0 \in P_+$ with $0 \in P_-$. Each point $z \in S$ can be considered as lying above its corresponding point $z \in P_+$ or $z \in P_-$ considered as a point in the complex plane. In this way, S is a Riemann surface spread over $\mathbb{C}$ with a natural projection map $S \to \mathbb{C}$ that is locally injective away from the so-called *branch point* $0 \in S$.

The surface S cannot be realized as an embedded surface in $\mathbb{R}^3$, but if one wants a model of S in $\mathbb{R}^3$ one has to allow a "self-intersection" of S along the branch cut.

If $z \in S$ is arbitrary, then the point z lies in one of the copies P_+ or P_- (or both). If $z \in P_+$ or $z \in P_-$ and $z = re^{i\alpha}$, where $r \geq 0$ and $-\pi \leq \alpha \leq \pi$, we define $\sqrt{z} = \sqrt{r}e^{i\alpha/2}$ or $\sqrt{z} = -\sqrt{r}e^{i\alpha/2}$, respectively. This gives a well-defined square-root function on S.

11 The Riemann sphere

11.1. The extended complex plane. We want to compactify $\mathbb{C}$ by adding a point at infinity denoted by ∞. Then $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is called the *extended complex plane*. One can visualize $\widehat{\mathbb{C}}$ as the *Riemann sphere* by *stereographic projection*.

To do this, let $S = \{(u, v, w) \in \mathbb{R}^3 : u^2 + v^2 + w^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$. We consider the complex plane as a subset of $\mathbb{R}^3$ by the identification $z = x + iy \in \mathbb{C} \cong (x, y, 0) \in \mathbb{R}^3$, where $x, y \in \mathbb{R}$. Then the sphere S cuts $\mathbb{C}$ along its equator. Let $N = (0, 0, 1)$ be the "north pole" of S. Then one defines a projection map $\pi \colon S \setminus \{N\} \to \mathbb{C}$ as follows. If $Z \in S \setminus \{N\}$ is arbitrary, then we join N and Z by a line L. The line L intersects the complex plane $\mathbb{C} \subseteq \mathbb{R}^3$ in a unique point z. We define $\pi(Z) := z$. This *stereographic projection* π gives a bijection of $S \setminus \{N\}$ onto $\mathbb{C}$.

It is easy to find an explicit formula for the map π and its inverse. If $Z = (x_1, x_2, x_3) \in S \setminus \{N\}$ is arbitrary, then the line L passing through Z and N is given by

$$L = \{t(x_1, x_2, x_3) + (1 - t)(0, 0, 1) \in \mathbb{R}^3 : t \in \mathbb{R}\}.$$

The line L meets $\mathbb{C}$ at a point z determined by the equation

$$tx_3 + (1 - t) = 0.$$

Hence $t = 1/(1 - x_3)$. Inserting this back into the parametrization of L gives

$$Z = (x_1, x_2, x_3) \in S \setminus \{N\} \mapsto \pi(Z) = z = \frac{x_1 + ix_2}{1 - x_3} \in \mathbb{C}.$$

Conversely, let $z = x + iy \in \mathbb{C}$ be arbitrary, where $x, y \in \mathbb{R}$. The line L' passing through $N = (0, 0, 1)$ and $(x, y, 0) \cong z$ is given by

$$L' = \{t(x, y, 0) + (1 - t)(0, 0, 1) \in \mathbb{R}^3 : t \in \mathbb{R}\}.$$

A point on L' different from N lies on S iff

$$\begin{aligned} t^2x^2 + t^2y^2 + (1 - t)^2 = t^2|z|^2 + (1 - t)^2 &= 1 \\ \Rightarrow\ t^2(|z|^2 + 1) &= 2t \\ \Rightarrow\ t &= \frac{2}{|z|^2 + 1} \quad (t = 0 \text{ corresponds to } N). \end{aligned}$$

Hence

$$z = x + iy \in \mathbb{C} \mapsto \pi^{-1}(z) = \left(\frac{2x}{|z|^2+1}, \frac{2x}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} \right) \in S \setminus \{N\}.$$

If we make $|z|$ larger and larger, then the corresponding point $Z = \pi^{-1}(z)$ on S gets closer and closer to N. So it is natural to extend π to a bijection from S onto $\widehat{\mathbb{C}}$ by setting $\pi(N) = \infty$. In this way, we can identify $\widehat{\mathbb{C}}$ with S by stereographic projection and refers to $\widehat{\mathbb{C}}$ as the *Riemann sphere*.

Let $z, w \in \mathbb{C}$ be points in the complex plane, and $Z = \pi^{-1}(z)$ and $W = \pi^{-1}(w)$. Then for the Euclidean distance of Z and W we get

$$\begin{aligned}
|Z-W|^2 &= \left| \frac{2z}{|z|^2+1} - \frac{2w}{|w|^2+1} \right|^2 + \left(\frac{|z|^2-1}{|z|^2+1} - \frac{|w|^2-1}{|w|^2+1} \right)^2 \\
&= \frac{4|z(|w|^2+1) - w(|z|^2+1)|^2 + 4(|z|^2-|w|^2)^2}{(|z|^2+1)^2(|w|^2+1)^2} \\
&= -4\frac{z\overline{w} + w\overline{z}}{(|z|^2+1)(|w|^2+1)} + 4\frac{|z|^2(|w|^2+1)^2 + |w|^2(|z|^2+1)^2 + (|z|^2-|w|^2)^2}{(|z|^2+1)^2(|w|^2+1)^2} \\
&= -4\frac{z\overline{w} + w\overline{z}}{(|z|^2+1)(|w|^2+1)} + 4\frac{(|z|^2+|w|^2)(|z|^2+1)(|w|^2+1)}{(|z|^2+1)^2(|w|^2+1)^2} \\
&= 4\frac{|z|^2+|w|^2 - z\overline{w} - w\overline{z}}{(|z|^2+1)(|w|^2+1)} = 4\frac{|z-w|^2}{(|z|^2+1)(|w|^2+1)},
\end{aligned}$$

and so

$$|Z-W| = \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}.$$

Moreover,

$$|Z-N|^2 = \frac{4|z|^2}{(|z|^2+1)^2} + \frac{4}{(|z|^2+1)^2} = \frac{4}{(|z|^2+1)},$$

and so

$$|Z-N| = \frac{2}{\sqrt{1+|z|^2}}.$$

11.2. The chordal metric. We define a distance function on $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ as follows. For $z, w \in \mathbb{C} \subseteq \widehat{\mathbb{C}}$ we define

$$d(z,w) := \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}},$$

$$d(z,\infty) = d(\infty,z) := \frac{2}{\sqrt{1+|z|^2}},$$

and let $d(\infty,\infty) = 0$.

As we have seen above, d corresponds to the Euclidean metric on S by stereographic projection. This shows that d is a metric on $\widehat{\mathbb{C}}$, called the *chordal metric*. Unless otherwise stated, $\widehat{\mathbb{C}}$ will always carry this metric.

Stereographic projection is an isometry between S equipped with the (restriction of the) Euclidean metric on $\mathbb{R}^3$ and $\widehat{\mathbb{C}}$ equipped with the chordal metric. In particular, S and $\widehat{\mathbb{C}}$ are homeomorphic (i.e., there exists a continuous bijection between these spaces with continuous inverse). So the topological and metric properties of S and $\widehat{\mathbb{C}}$ are the same. In particular, $\widehat{\mathbb{C}}$ is a compact metric space.

Remark 11.3. It is not hard to show that $\mathbb{C}$ equipped with the restriction of the chordal metric and $\mathbb{C}$ equipped with the usual Euclidean metric have the same topology (i.e., the same open sets). So chordal and Euclidean metrics are *equivalent metrics*.

In particular, if $\{z_n\}$ is a sequence in $\mathbb{C}$ and $z \in \mathbb{C}$, then $z_n \to z$ with respect to the chordal metric if and only if $z_n \to z$ with respect to the Euclidean metric.

We have $z_n \to \infty \in \widehat{\mathbb{C}}$ (convergence in the metric space $\widehat{\mathbb{C}}$) iff

$$d(z_n,\infty) = \frac{2}{\sqrt{1+|z_n|^2}} \to 0$$

iff $|z_n| \to +\infty$ (in the sense of real analysis).

Definition 11.4. Let X be a topological Hausdorff space (e.g., a metric space), and $n \in \mathbb{N}$. Then X is called a *(topological) n-manifold* if every point $x \in X$ has an open neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$.

A *chart* on an n-manifold X is a homeomorphism $\varphi\colon U \to V$ of an open subset $U \subseteq X$ onto an open subset $V \subseteq \mathbb{R}^n$. An *atlas* on X is a family $\mathcal{A} = \{\varphi_i\colon U_i \to V_i : i \in I\}$ of charts on X such that $X = \bigcup_{i\in I} U_i$, i.e., the chart neighborhoods of charts in the family cover the whole manifold X.

Recall that a topological space X is called *Hausdorff* if for all points $x, y \in X$, $x \neq y$, there exists open sets $U, V \subseteq X$ with $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

Our definition of an n-manifold is slightly non-standard as one usually requires that an n-manifold should have a *countable basis* for its topology (i.e., there exists a countable set $\mathcal{B}$ of open sets in X such that every open set in U is a union of sets in $\mathcal{B}$).

We are mostly interested in the case $n = 2$. A 2-manifold is also called a *surface*. If we make the identification $\mathbb{R}^2 \cong \mathbb{C}$, then we can think of a chart on a surface as a map into $\mathbb{C}$.

Example 11.5. The unit sphere $S \subseteq \mathbb{R}^3$ (and hence also $\widehat{\mathbb{C}}$) is a topological 2-manifold. To see this, it is enough to exhibit an atlas of charts on S. It consists of two charts, namely the map $\varphi_1 \colon S \setminus \{n\} \to \mathbb{R}^2$ given by stereographic projection from the north pole $n = (0, 0, 1)$, and the map $\varphi_2 \colon S \setminus \{s\} \to \mathbb{R}^2$ given by stereographic projection from the south pole $s = (0, 0, -1)$ (defined in the obvious way).

Similarly, one can define an atlas on $\widehat{\mathbb{C}}$ given by two charts ψ_1 and ψ_2. Here $\psi_1 \colon U_1 := \mathbb{C} \subseteq \widehat{\mathbb{C}} \to \mathbb{C}$, $\psi_1(z) = z$ for $z \in \mathbb{C}$, and $\psi_2 \colon U_2 := \mathbb{C} \setminus \{0\} \cup \{\infty\} \subseteq \widehat{\mathbb{C}} \to \mathbb{C}$,

$$\psi_2(z) = \begin{cases} 1/z & \text{for } z \in \mathbb{C} \setminus \{0\}, \\ \\ 0 & \text{for } z = \infty. \end{cases}$$

Definition 11.6. Let $U, V \subseteq \mathbb{C}$ be open sets. A map $f \colon U \to V$ is called a *biholomorhism* if f is a bijection, f is holomorphic on U, and f^{-1} is holomorphic on V.

Two charts $\varphi_1 \colon U_1 \to V_1 \subseteq \mathbb{C}$ and $\varphi_2 \colon U_2 \to V_2$ on a 2-manifold X are called *holomorphically compatible* if $U_1 \cap U_2 = \emptyset$, or if $U_1 \cap U_2 \neq \emptyset$ and the map

$$\varphi_2 \circ \varphi_1^{-1} | \varphi_1(U_1 \cap U_2) \colon \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

is a biholomorphism.

Example 11.7. Let $\psi_1 \colon U_1 \to \mathbb{C}$ and $\psi_2 \colon U_2 \to \mathbb{C}$ be the charts on $\widehat{\mathbb{C}}$ as in Example 11.5. Then ψ_1 and ψ_2 are holomorphically compatible. Indeed, $U_1 = \mathbb{C}$ and $U_2 = \mathbb{C} \setminus \{0\} \cup \{\infty\}$, and so $U_1 \cap U_2 = \mathbb{C} \setminus \{0\}$. Then $W := \psi_1(U_1 \cap U_2) = \psi_2(U_1 \cap U_2) = \mathbb{C} \setminus \{0\}$. Moreover, $f := \psi_2 \circ \psi_1^{-1} | W \colon W \to W$ is given by $f(z) = 1/z$ for $z \in W$. This is a holomorphic bijection with holomorphic inverse $z \mapsto f^{-1}(z) = 1/z$.

Definition 11.8. An atlas $\mathcal{A} = \{\varphi_i \colon U_i \to V_i \subseteq \mathbb{C} : i \in I\}$ on a 2-manifold X is called a *complex atlas* if the charts in $\mathcal{A}$ are pairwise holomorphically compatible.

Two complex atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ on X are called *analytically equivalent* if each chart in $\mathcal{A}_1$ is holomorphically compatible with each chart in $\mathcal{A}_2$. This defines an equivalence relation for atlases. A *complex structure* on X is an equivalence class of analytically equivalent atlases on X. A *Riemann surface* is a topological 2-manifold equipped with a complex structure.

Remark 11.9. One should think of the complex structure on a Riemann surface as being represented by some fixed complex atlas. A *complex chart* on a Riemann surface is a chart that is holomorphically compatible with all the charts in one atlas (and hence in all atlases) that define the complex structure. Often it is convenient to add complex charts to a given atlas as needed, for example if one wants to make the chart neighborhoods smaller: if $\varphi\colon U \to V$ is a chart in a complex atlas $\mathcal{A}$, and $U' \subseteq U$ is open, then $\varphi|U'\colon U' \to \varphi(U')$ is a chart that is holomorphically compatible with each chart in $\mathcal{A}$.

Definition 11.10. Let X be a Riemann surface, and $f\colon X \to \mathbb{C}$. Then f is called a *holomorphic function* if for every complex chart $\varphi\colon U \to V \subseteq \mathbb{C}$ on X the function $f \circ \varphi^{-1}\colon V \to \mathbb{C}$ is a holomorphic function on the open set $V \subseteq \mathbb{C}$.

Definition 11.11. Let X and Y be Riemann surfaces, and $f\colon X \to Y$ be continuous. Then f is called a *holomorphic map* (between X and Y) if for every complex chart $\varphi\colon U \to V \subseteq \mathbb{C}$ on X and every complex chart $\psi\colon U' \to V' \subseteq \mathbb{C}$ on Y with $f(U) \subseteq U'$ the function $\psi \circ f \circ \varphi^{-1} : V \to V'$ is holomorphic.

Note that the function $\psi \circ f \circ \varphi^{-1}$ is defined on the open set $V \subseteq \mathbb{C}$ and takes values in $V' \subseteq \mathbb{C}$. So it makes sense to require the holomorphicity of this function (as defined in Section 3).

Example 11.12. The atlas $\mathcal{A}$ consisting of the charts ψ_1 and ψ_2 on the Riemann sphere $\widehat{\mathbb{C}}$ as defined Example 11.5 is a complex atlas (see Example 11.7). In the following, we always think of $\widehat{\mathbb{C}}$ as a Riemann surface equipped with the complex structure represented by the complex atlas $\mathcal{A}$.

How do we verify that a given continuous map $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is holomorphic? We check this locally near each point $z_0 \in \widehat{\mathbb{C}}$. Let $w_0 := f(z_0)$. The basic idea is this: whenever the point ∞ is involved, we send it to 0 by the chart ψ_2. This leads to several cases.

Case 1: $z_0, w_0 \in \mathbb{C}$. In this case, $z \mapsto f(z)$ is just a complex-valued function near z_0, and we check its holomorphicity as usual.

Case 2: $z_0 = \infty$, $w_0 \in \mathbb{C}$. We check $z \mapsto g(z) := f(1/z)$ for holomorphicity near 0.

Here it is understood that $g(0) := f(\infty)$, and we make similar definitions in the remaining two cases.

Case 3: $z_0 \in \mathbb{C}$, $w_0 = \infty$. We check $z \mapsto 1/f(z)$ for holomorphicity near $z_0 \in \mathbb{C}$.

Case 4: $z_0 = w_0 = \infty$. We check $z \mapsto 1/f(1/z)$ for holomorphicity near 0.

Example 11.13. Every rational function R can be considered as a holomorphic map $R\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. To see this, let $R = P/Q$, where P and Q are polynomials with no common roots and $Q \neq 0$. We write

$$\begin{aligned} P(z) &= a_0 + a_1 z + \cdots + a_n z^n, \quad a_n \neq 0, \\ Q(z) &= b_0 + b_1 z + \cdots + b_k z^k, \quad b_k \neq 0. \end{aligned}$$

Then R is given by $R(z) = P(z)/Q(z)$ for $z \in \mathbb{C}$ if $Q(z) \neq 0$. We extend R continuously to a map $R\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. To do this, we set

$$R(z) := \lim_{w \to z} R(w) = \infty$$

for $z \in \mathbb{C}$ if $Q(z) = 0$ (note that then $P(z) \neq 0$) and

$$R(\infty) := \lim_{w \to \infty} R(w) = \begin{cases} 0 & \text{if } n < k, \\ a_n/b_k & \text{if } n = k, \\ \infty & \text{if } n > k. \end{cases}$$

Then $R(z)$ is defined for all $z \in \widehat{\mathbb{C}}$, and $R\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is continuous.

Moreover, $R\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a holomorphic map as can be seen by the case analysis as in Example 11.12.

For example, suppose $z_0 \in \mathbb{C}$ and $Q(z_0) = 0$. Then $R(z_0) = \infty$, and so we have to investigate the holomorphicity of

$$z \mapsto \frac{1}{R(z)} = \frac{Q(z)}{P(z)} \tag{27}$$

near z_0. Since P and Q have no common roots, we have $P(z_0) \neq 0$ and so $P(z) \neq 0$ for z near z_0. The holomorphicity of the function in (27) near z_0 follows.

12 Möbius transformations

Definition 12.1. A rational map $R\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of the form

$$R(z) = \frac{az+b}{cz+d} \quad \text{with } a,b,c,d \in \mathbb{C},\ ad-bc \neq 0, \tag{28}$$

is called a *Möbius transformation*. The set of all Möbius transformations is denoted by Möb.

The condition $ad - bc \neq 0$ in (28) ensures that we cannot cancel the numerator against denominator in the formula for $R(z)$. Note that $ad - bc$ is the determinant of the coefficient matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ associated with R.

12.2. Composition of Möbius transformations. The composition of two Möbius transformations is again a Möbius transformation. This operation corresponds to multiplication of the associated coefficient matrices. Indeed, suppose that $S, T \in$ Möb, where

$$S(w) = \frac{a_{11}w + a_{12}}{a_{21}w + a_{22}}, \quad \text{and} \quad w = T(z) = \frac{b_{11}z + b_{12}}{b_{21}z + b_{22}}.$$

Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

be the corresponding coefficient matrices. Then $S \circ T$ is also a Möbius transformation, because

$$\begin{aligned}(S \circ T)(z) = S(T(z)) &= \frac{a_{11}\dfrac{b_{11}z + b_{12}}{b_{21}z + b_{22}} + a_{12}}{a_{21}\dfrac{b_{11}z + b_{12}}{b_{21}z + b_{22}} + a_{22}} \\ &= \frac{(a_{11}b_{11} + a_{12}b_{21})z + a_{11}b_{12} + a_{12}b_{22}}{(a_{21}b_{11} + a_{22}b_{21})z + a_{21}b_{12} + a_{22}b_{22}},\end{aligned}$$

and its associated coefficient matrix is

$$\begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix} = AB,$$

which is the product of A and B.

12.3. Möb **is a group.** The set Möb of Möbius transformations with the composition given by composition $\circ$ of maps is a group. Indeed:

1. If $S, T \in$ Möb, then $S \circ T \in$ Möb.

2. Associativity of $\circ$ is clear, because composition of maps is associative.

3. The identity on $\widehat{\mathbb{C}}$ is the unit element in Möb; note that $\mathrm{id}_{\widehat{\mathbb{C}}} \in$ Möb, because
$$\mathrm{id}_{\widehat{\mathbb{C}}}(z) = z = \frac{1 \cdot z + 0}{0 \cdot z + 1}.$$

4. If $S \in$ Möb with associated coefficient matrix A, then the Möbius transformation with associated coefficient matrix A^{-1} (it exists, because $\det(A) \neq 0$) is the inverse S^{-1} of S.

In particular, every Möbius transformation S is a biholomorphism from $\widehat{\mathbb{C}}$ onto $\widehat{\mathbb{C}}$ (i.e., S is a bijection, and both maps S and S^{-1} are holomorphic).

12.4. Generators of Möb**.** The group Möb is *generated* by translations, dilations, rotations, and the *inversion*. This means that every $R \in$ Möb can be written as a composition of maps of these types. These maps are Möbius transformation of the following forms:

$$\begin{aligned}
&S(z) = z + a, \quad a \in \mathbb{C}, \quad \text{(translation)},\\
&S(z) = \lambda z, \quad \lambda > 0, \quad \text{(dilation)},\\
&S(z) = e^{i\theta} z, \quad \theta \in \mathbb{R}, \quad \text{(rotation)},\\
&S(z) = 1/z \quad \text{(inversion)}.
\end{aligned}$$

To see that Möb is generated by these maps, let R be a Möbius transformation of the form
$$R(z) = \frac{az + b}{cz + d}.$$
Assume $c \neq 0$ (the case $c = 0$ is similar and easier). By dividing the numerator in the fraction representing R by the denominator, we can find $A, B \in \mathbb{C}$, $B \neq 0$, such that
$$R(z) = A + \frac{B}{cz + d}.$$

Then the map $z \mapsto R(z)$ can be written as a composition of generators as above in the following way:

$$\begin{aligned} z &\overset{\text{dilation}}{\longrightarrow} |c|z \overset{\text{rotation}}{\longrightarrow} cz \overset{\text{translation}}{\longrightarrow} cz+d \overset{\text{inversion}}{\longrightarrow} \frac{1}{cz+d} \\ &\overset{\text{dilation+rotation}}{\longrightarrow} \frac{B}{cz+d} \overset{\text{translation}}{\longrightarrow} A + \frac{B}{cz+d} = R(z). \end{aligned}$$

12.5. Fixed points of Möbius transformations. Every Möbius transformation $R \neq \mathrm{id}_{\widehat{\mathbb{C}}}$ has precisely one or two fixed points. To see this, let R be a Möbius transformation of the form

$$R(z) = \frac{az+b}{cz+d}.$$

We consider the fixed point equation

$$z = \frac{az+b}{cz+d}. \tag{29}$$

This leads to two cases.

Case 1: $c = 0$. Then $d \neq 0$ and (29) is equivalent to

$$\frac{a}{d}z + \frac{b}{d} = z \Leftrightarrow \left(\frac{a}{d} - 1\right) z + \frac{b}{d} = 0.$$

This equation has exactly one solution $z \in \mathbb{C}$ if $a/d - 1 \neq 0$, and no solution $z \in \mathbb{C}$ if $a/d - 1 = 0$ and $b/d \neq 0$. Note that if $a/d - 1 = 0$ and $b/d = 0$, then $a = d$ and $b = c = 0$. This is impossible as $R \neq \mathrm{id}_{\widehat{\mathbb{C}}}$.

So R has no or exactly one fixed point $z \in \mathbb{C}$. Since $c = 0$, we must have $a \neq 0$, and so $R(\infty) = \infty$. Hence R has precisely one or two fixed points in $\widehat{\mathbb{C}}$.

Case 2: $c \neq 0$. Then $R(\infty) = a/c \neq \infty$. So ∞ is not a fixed point of R. The fixed point equation (29) is equivalent to

$$cz^2 + (d-a)z + b = 0.$$

Since $c \neq 0$, this equation has precisely one or two solutions $z \in \mathbb{C}$. It follows that R has precisely one or two fixed points in $\widehat{\mathbb{C}}$.

12.6. Möbius transformations are uniquely determined by images of three distinct points. Let $S, T \in$ Möb, and $z_1, z_2, z_3 \in \widehat{\mathbb{C}}$ be three distinct points. If $S(z_k) = T(z_k)$ for $k = 1, 2, 3$, then $S = T$.

Indeed, if $S(z_k) = T(z_k)$, then $(T^{-1} \circ S)(z_k) = z_k$ for $k = 1, 2, 3$. So the Möbius transformation $T^{-1} \circ S$ has three distinct fixed points. By what we have seen in Subsection 12.5, this is only possible if $T^{-1} \circ S = \mathrm{id}_{\widehat{\mathbb{C}}}$. Hence $S = T$.

12.7. Cross-ratios. If $z_1, z_2, z_2, z_4 \in \widehat{\mathbb{C}}$ are four distinct points, we define their *cross-ratio* as

$$(z_1, z_2, z_3, z_4) := \frac{z_1 - z_3}{z_1 - z_4} : \frac{z_2 - z_3}{z_2 - z_4} = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}.$$

This has to interpreted as an appropriate limit if one of the points is equal to $\infty \in \widehat{\mathbb{C}}$; for example,

$$(\infty, z_2, z_3, z_4) := \lim_{z \to \infty} (z, z_2, z_3, z_4) = \frac{z_2 - z_4}{z_2 - z_3}.$$

Note that the map

$$z \mapsto (z, z_2, z_3, z_4) = \frac{(z - z_3)(z_2 - z_4)}{(z - z_4)(z_2 - z_3)}$$

is a Möbius transformation. The values of this Möbius transformation for $z = z_2, z_3, z_4$ have to be interpreted as appropiate limts. Hence this Möbius transformation maps z_2 to 1, z_3 to 0, and z_4 to ∞.

12.8. Möbius transformations preserve cross-ratios. If $S \in$ Möb and $z_1, z_2, z_3, z_4 \in \widehat{\mathbb{C}}$ are distinct points, then

$$(S(z_1), S(z_2), S(z_3), S(z_4)) = (z_1, z_2, z_3, z_4).$$

Proof. The maps $z \mapsto (z, z_2, z_3, z_4)$ and $z \mapsto (S(z), S(z_2), S(z_3), S(z_4))$ are Möbius transformations. The first maps z_2 to 1, z_3 to 0, and z_3 to ∞. The second map has the same behavior as is indicated by the following table:

$$\begin{array}{llll}
z & \mapsto w = S(z) & \mapsto & (w, S(z_2), S(z_3), S(z_4)), \\
z_2 & \mapsto \quad S(z_2) & \mapsto & \quad 1, \\
z_3 & \mapsto \quad S(z_3) & \mapsto & \quad 0, \\
z_4 & \mapsto \quad S(z_4) & \mapsto & \quad \infty.
\end{array}$$

So these Möbius transformations agree on z_2, z_3, z_4. It follows that they are the same (see 12.6), and so give the same value for $z = z_1$. $\square$

12.9. Möb **acts triply transitive on $\widehat{\mathbb{C}}$.** If $z_1, z_2, z_3 \in \widehat{\mathbb{C}}$ are three distinct points, and $w_1, w_2, w_3 \in \widehat{\mathbb{C}}$ are three distinct points, then there exists a unique Möbius transformation S with $S(z_k) = w_k$ for $k = 1, 2, 3$.

Proof. Uniqueness follows from 12.6.

To establish existence, let $U, V \in$ Möb be given by

$$U(z) = (z, z_1, z_2, z_3) \quad \text{and} \quad V(z) = (z, w_1, w_2, w_3)$$

for $z \in \widehat{\mathbb{C}}$. These maps have mapping properties as indicated by the following table:

$$\begin{array}{ccccccc} & U & & \qquad & & V & \\ z_1 & \mapsto & 1, & & w_1 & \mapsto & 1, \\ z_2 & \mapsto & 0, & & w_2 & \mapsto & 0, \\ z_3 & \mapsto & \infty, & & w_3 & \mapsto & \infty. \end{array}$$

Then $S := V^{-1} \circ U \in$ Möb has the desired mapping behavior:

$$\begin{array}{ccccc} & U & & V^{-1} & \\ z_1 & \mapsto & 1 & \mapsto & w_1, \\ z_2 & \mapsto & 0 & \mapsto & w_2, \\ z_3 & \mapsto & \infty & \mapsto & w_3. \end{array}$$

□

The map S in the last proof satisfies $V(S(z)) = U(z)$, i.e.,

$$(S(z), w_2, w_3, w_4) = (z, z_2, z_3, z_4).$$

This equation can be used to compute $S(z)$ explicitly.

Example 12.10. How to find the Möbius transformation S that maps 1, i, $-i$, to -1, 1, ∞, respectively?

Solution: The map $z \mapsto w = S(z)$ satisfies

$$(z, 1, i, -i) = (w, -1, 1, \infty),$$

or equivalently,

$$\frac{(z-i)(1+i)}{(z+i)(1-i)} = \text{“}\frac{(w-1)(-1-\infty)}{(w-\infty)(-1-1)}\text{”} = \frac{1-w}{2}.$$

Of course, the term in the middle enclosed in "..." has a purley symbolic meaning and has to be interpreted as an appropriate limit given by the last term. Solving for w gives (after some computation)

$$w = S(z) = \frac{-(1+2i)z + (-1+2i)}{z+1}.$$

12.11. Circles in $\widehat{\mathbb{C}}$. A *circle* (in $\widehat{\mathbb{C}}$) is either a Euclidean circle in $\mathbb{C}$ in the usual sense or a line ("circle through ∞"). Three distinct points in $\widehat{\mathbb{C}}$ determine a unique circle on which they lie.

The circles in $\widehat{\mathbb{C}}$ are precisely the curves whose points $z \in \mathbb{C}$ can be characterized by an equation of the form

$$\alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0, \tag{30}$$

where $\alpha, \gamma \in \mathbb{R}$, $\beta \in \mathbb{C}$, and $|\beta|^2 > \alpha\gamma$.

To see this, note that every equation of the form (30) represents a circle. Indeed, there are two cases.

Case 1. $\alpha = 0$. Then $|\beta|^2 > \alpha\gamma = 0$, and so $\beta \neq 0$. Moreover, if $\beta = a+ib$ and $z = x + iy$, where $a, b, x, y \in \mathbb{R}$, then a and b cannot both be equal to 0, and (30) is equivalent to

$$2ax - 2by + \gamma = 0.$$

This equation represents a line.

Case 2. $\alpha \neq 0$. Then $\alpha = 1$ without loss of generality, and (30) can be written as

$$\begin{aligned} 0 = z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma &= (z+\bar{\beta})(\bar{z}+\beta) + \gamma - |\beta|^2 \\ \Leftrightarrow |z - (-\bar{\beta})|^2 &= |\beta|^2 - \gamma > 0. \end{aligned}$$

This represents a circle of radius $R = (|\beta|^2 - \gamma)^{1/2}$ centered at $-\bar{\beta}$.

Conversely, every equation for a circle or a line in $\mathbb{C}$ can be given the form (30) (exercise!).

12.12. Möbius transformations map circles to circles. If $S \in$ Möb, and $C \subseteq \widehat{\mathbb{C}}$ is a circle, then $S(C) \subseteq \widehat{\mathbb{C}}$ is a circle.

Proof. Since we know that the group Möb is generated by translations, dilations, rotations, and the inversion $z \mapsto I(z) := 1/z$ (see 12.4), it is enough to show the statement for these types of maps.

Since the assertion is clear for translations, dilations, and rotations, it suffices to prove it for the inversion I. Let $C \subseteq \widehat{\mathbb{C}}$ be a circle, and suppose it it represented by an equation of the form (30). Then

$$\begin{aligned} z \in C &\Leftrightarrow \alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0 \\ &\Leftrightarrow \alpha + \frac{\beta}{\bar{z}} + \frac{\bar{\beta}}{z} + \frac{\gamma}{z\bar{z}} = 0 \\ &\Leftrightarrow \alpha + \bar{\beta}w + \beta\bar{w} + \gamma w\bar{w} = 0 \quad \text{with } w = I(z) = 1/z, \\ &\Leftrightarrow w \in \widetilde{C}, \end{aligned}$$

where $\widetilde{C} \subseteq \widehat{\mathbb{C}}$ is the circle represented by the equation

$$\gamma w\bar{w} + \bar{\beta}w + \beta\bar{w} + \alpha = 0.$$

This is equivalent to $I(C) = \widetilde{C}$. □

12.13. Cross-ratios and points on a circle. Four distinct points z_1, z_2, z_3, z_4 in $\widehat{\mathbb{C}}$ lie on a circle if and only if $(z_1, z_2, z_3, z_4) \in \mathbb{R}$.

Proof. Let C be the unique circle with $z_2, z_3, z_4 \in C$, and $S \in$ Möb be given by $S(z) = (z, z_2, z_3, z_4)$. Then $S(C)$ is a circle containing $S(z_2) = 1$, $S(z_2) = 0$, and $S(z_4) = \infty$. Hence $S(C) = \mathbb{R} \cup \{\infty\} =: \widehat{\mathbb{R}}$. Note that for $u \in \mathbb{C}$ we have

$$(u, 1, 0, \infty) = \text{“}\frac{(u-0)(1-\infty)}{(u-\infty)(1-0)}\text{”} = u.$$

Using this and the invariance of cross-ratios under Möbius transformations, we see that z_1, z_2, z_3, z_4 lie on a circle iff $z_1 \in C$ iff $S(z_1) \in S(C) = \widehat{\mathbb{R}}$ iff

$$S(z_1) = (S(z_1), 1, 0, \infty)) = (S(z_1), S(z_2), S(z_3), S(z_4)) \in \mathbb{R}$$

iff $(z_1, z_2, z_3, z_4) \in \mathbb{R}$. □

Lemma 12.14. *Let $S \in$ Möb. Then $S(\widehat{\mathbb{R}}) = \widehat{\mathbb{R}}$ if and only if S has the form*

$$S(z) = \frac{az+b}{cz+d}, \quad \textit{where } a, b, c, d \in \mathbb{R},\ ad - bc \neq 0. \tag{31}$$

Proof. $\Leftarrow$: If S can be written in the form (31), then S maps point in $\widehat{\mathbb{R}}$ to points in $\widehat{\mathbb{R}}$. This is only possible if $S(\widehat{\mathbb{R}}) = \widehat{\mathbb{R}}$.

$\Rightarrow$: Conversely, suppose S can be written as in (31), let $z \in \widehat{\mathbb{C}}$, and $w = S(z)$. Then

$$z = (z, 1, 0, \infty) = (w, S(1), S(0), S(\infty)) = \frac{\alpha w + \beta}{\gamma w + \delta},$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, $\alpha\delta - \beta\gamma \neq 0$. Since $S(1), S(0), S(\infty) \in \widehat{\mathbb{R}}$, we actually have $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. Solving this for $w = S(z)$ in terms of z, we get an expression for $S(z)$ as in (31). □

Lemma 12.15. *Let $S \in$ Möb with $S(\widehat{\mathbb{R}}) = \widehat{\mathbb{R}}$. Then $S(\bar{z}) = \overline{S(z)}$ for $z \in \widehat{\mathbb{C}}$ (here $\overline{\infty} := \infty$).*

Proof. This immediately follows from Lemma 12.14. □

12.16. Points symmetric with respect to a circle. Let $C \subseteq \widehat{\mathbb{C}}$ be a circle, and $z, z^* \in \widehat{\mathbb{C}}$. We say that z and z^* are *symmetric with respect to C* if the following condition is true: Pick $U \in$ Möb such that $U(C) = \widehat{\mathbb{R}}$. Then the requirement is that

$$U(z^*) = \overline{U(z)} \quad \text{which is equivalent to} \quad U(z) = \overline{U(z^*)}. \tag{32}$$

Note that a Möbius transformation U with $U(C) = \widehat{\mathbb{R}}$ exists, because we can send three distinct points on C to $0, 1, \infty$ by a Möbius transformation. Then $U(C) = \widehat{\mathbb{R}}$.

Note also that condition (32) is independent of the Möbius transformation we choose. To see this, suppose that V is another Möbius transformation with $V(C) = \widehat{\mathbb{R}}$. If $S := V \circ U^{-1} \in$ Möb, then $S(\widehat{\mathbb{R}}) = \widehat{\mathbb{R}}$, and so $S(\bar{w}) = \overline{S(w)}$ for all $w \in \widehat{\mathbb{C}}$. So if (32) is true, then we also have

$$\begin{aligned} V(z^*) &= (V \circ U^{-1})(U(z^*)) = S(\overline{U(z)}) \\ &= \overline{S(U(z))} = \overline{V(z)}. \end{aligned}$$

12.17. The symmetry principle. Let $S \in$ Möb, and z and z^* be points in $\widehat{\mathbb{C}}$ that are symmetric with respect to a circle $C \subseteq \widehat{\mathbb{C}}$. Then $w = S(z)$ and $w^* = S(z^*)$ are symmetric with respect to the circle $\widetilde{C} = S(C)$.

Proof. Pick $U \in$ Möb such that $U(C) = \widehat{\mathbb{R}}$. Then $U(z^*) = \overline{U(z)}$. Then for $W := U \circ S^{-1} \in$ Möb we have $W(\widetilde{C}) = U(C) = \widehat{\mathbb{R}}$. Moreover,

$$\begin{aligned} W(w^*) &= (U \circ S^{-1})(S(z^*)) = U(z^*) \\ &= \overline{U(z)} = \overline{(U \circ S^{-1})(S(z))} = \overline{W(w)}. \end{aligned}$$

Hence w and w^* are symmetric with respect to $\widetilde{C}$. □

Lemma 12.18. *Let $C \subseteq \widehat{\mathbb{C}}$ be a circle, and $z_1, z_2, z_3 \in C$ be distinct points. Then $z \in \widehat{\mathbb{C}}$ and $z^* \in \widehat{\mathbb{C}}$ are symmetric with respect to C if and only if*

$$(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}. \tag{33}$$

Proof. Note that $u \mapsto (u, z_1, z_2, z_3)$ is a Möbius transformation that maps C onto $\widehat{\mathbb{R}}$ (cf. 12.13). So (33) corresponds to (32). □

12.19. Geometric description of symmetry with respect to a circle. Let $C \subseteq \widehat{\mathbb{C}}$ be a circle, and z and z^* be points that are symmetric with respect to C.

Case 1: C is a line L. Then we can find a Möbius transformation U that is a composition of a rotation and a translation that maps L to $\widehat{\mathbb{R}}$. If $w = U(z)$ and $w^* = U(z^*)$, then $w^* = \bar{w}$, and so w^* is the image of w under reflection in $\mathbb{R}$. Since U that is a composition of a rotation and a translation, this reflections corresponds to the reflection in the line L. It follows that z^* is the image of z under reflection in L.

Since we can reverse the argument, it follows that two points z^* and z are symmetric with respect to the line $L = C$ if and only if z^* is the image of z under reflection in L.

Case 2: $C \subseteq \mathbb{C}$ is a Euclidean circle. Then a point $w \in C$ satisfies an equation of the type

$$|w - a| = R \;\Leftrightarrow\; (w-a)\overline{(w-a)} = R^2 \;\Leftrightarrow\; \overline{w-a} = \frac{R^2}{w-a}, \tag{34}$$

where $a \in \mathbb{C}$ is the center and $R > 0$ is the radius of C.

We pick three distinct points $z_1, z_2, z_3 \in C$. Then from Lemma 12.18 and the invariance of cross-ratios under Möbius transformations we conclude that

$$\begin{aligned}
(z^*, z_1, z_2, z_3) &= \overline{(z, z_1, z_2, z_3)} = (\overline{z}, \overline{z_1}, \overline{z_2}, \overline{z_3}) \\
&= (\overline{z-a}, \overline{z_1-a}, \overline{z_2-a}, \overline{z_3-a}) \quad \text{(by applying } u \mapsto u - \bar{a}) \\
&= \left(\overline{z-a}, \frac{R^2}{z_1-a}, \frac{R^2}{z_2-a}, \frac{R^2}{z_3-a}\right) \quad \text{(by (34))} \\
&= \left(\frac{R^2}{\bar{z}-\bar{a}}, z_1 - a, z_2 - a, z_3 - a\right) \quad \text{(by applying } u \mapsto R^2/u) \\
&= \left(a + \frac{R^2}{z-a}, z_1, z_2, z_3\right) \quad \text{(by applying } u \mapsto u + a).
\end{aligned}$$

If we consider the cross-ratio as a function of the first argument with the other arguments fixed, then it is a Möbius transformation and hence a bijection. So we conclude that

$$z^* = a + \frac{R^2}{\bar{z} - \bar{a}} \quad \text{(where } z^* = \infty \text{ if } z = a\text{, and } z^* = a \text{ if } z = \infty\text{).} \tag{35}$$

This equation defines a map $z \mapsto z^*$ on $\widehat{\mathbb{C}}$ called the *reflection* in the circle C.

If $z = a + re^{it}$ with $r > 0$ and $t \in \mathbb{R}$, then

$$z^* = a + \frac{R^2}{re^{-it}} = a + \frac{R^2}{r}e^{it}.$$

This implies that z and z^* lie one the ray $S = \{a + \rho e^{it} : \rho > 0\}$ emanating from a, and we have

$$|z - a| \cdot |z^* - a| = R^2.$$

Note that the fixed points of the reflection in the circle C are precisely the points in C.

Again one can reverse the argument, and our discussion shows that two points z and z^* are symmetric with respect to C if and only if z^* is the image of z under the reflection in C.

12.20. Oriented circles. We say that a circle $C \subseteq \widehat{\mathbb{C}}$ is *oriented* if we specify an ordered triple (z_1, z_2, z_3) of distinct points on C. Given an orientation on C we define the *right side* of C as the set

$$\{z \in \widehat{\mathbb{C}} : \operatorname{Im}(z, z_1, z_2, z_3) > 0\}$$

and the *left side* of C as the set

$$\{z \in \widehat{\mathbb{C}} : \operatorname{Im}(z, z_1, z_2, z_3) < 0\}.$$

Note that $\operatorname{Im}(z, z_1, z_2, z_3) = 0$ if and only if $z \in C$. This implies that the left and the right side of C form the two connected components of the complement of C in $\widehat{\mathbb{C}}$.

12.21. Möbius transformations are orientation-preserving. Let $S \in$ Möb and $C \subseteq \widehat{\mathbb{C}}$ be a circle. Let $\widetilde{C} = S(C)$ be the image of C under S. Suppose C is oriented by the three distinct points $z_1, z_2, z_3 \in C$. We orient

$\widetilde{C}$ by the image points $S(z_1), S(z_2), S(z_3) \in \widetilde{C}$. Then S takes the left side of C to the left side of $\widetilde{C}$, and consequently the right side of C to the right side of $\widetilde{C}$.

Indeed, a point $z \in \widehat{\mathbb{C}}$ belongs to the left side of C if and only if

$$\operatorname{Im}(z, z_1, z_2, z_3) = \operatorname{Im}(S(z), S(z_1), S(z_2), S(z_3)) < 0.$$

This is true if and only if $S(z)$ belongs to the left side of $\widetilde{C}$.

12.22. Conformal maps. Let $U, V \subseteq \widehat{\mathbb{C}}$ be open, and $f: U \to V$ be a map. Then f is called a *conformal map* (from U onto V) if f is a bijection, f is holomorphic on U, and f^{-1} is holomorphic on V. In other words, a conformal map from U onto V is the same as a biholomorphism from U onto V.

If $f: U \to V$ is a conformal map, and $g = f^{-1}$, then $g(f(z)) = z$ for all $z \in U$. In particular, if $U, V \subseteq \mathbb{C}$, then

$$g'(f(z)) \cdot f'(z) = 1$$

and so $f'(z) \neq 0$ for all $z \in U$. So the derivative of a conformal map does not vanish anywhere (this also follows from Theorem 9.3).

12.23. Conformal maps preserve angles between curves. Let $\Omega \subseteq \mathbb{C}$ be open, and $\gamma_1, \gamma_2 : [a, b] \to \Omega$ be two smooth paths with common initial point $p = \gamma_1(a) = \gamma_2(a)$. If $\gamma_1'(a), \gamma_2'(a) \neq 0$, then we define the *angle between γ_1 and γ_2 at p* as

$$\angle_p(\gamma_1, \gamma_2) := \arg \gamma_2'(a) - \arg \gamma_1'(a).$$

This is equal to the angle between the tangent vectors of γ_1 and γ_2 at p. Intuitively, $\angle_p(\gamma_1, \gamma_2)$ is the angle of a rotation at p that sends the direction of $\gamma_1'(a)$ into the direction of $\gamma_2'(a)$. Note that the order of the paths γ_1 and γ_2 is important here.

Now let $f: \Omega \to \mathbb{C}$ be a holomorphic function with $f'(p) \neq 0$. Define $q := f(p)$ and $\sigma_k := f \circ \gamma_k$, $k = 1, 2$. Then

$$\angle_q(\sigma_1, \sigma_2) = \angle_p(\gamma_1, \gamma_2). \tag{36}$$

To see this, assume that $f'(p) = re^{i\alpha}$. Then

$$\sigma_k'(a) = f'(\gamma_k(a)) \cdot \gamma_k'(a) = f'(p) \cdot \gamma_k'(a) = re^{i\alpha}\gamma_k'(a),$$

and so

$$\arg\sigma_k'(a) = \alpha + \arg\gamma_k'(a)$$

for $k = 1, 2$. Hence

$$\arg\sigma_2'(a) - \arg\sigma_1'(a) = \arg\gamma_2'(a) - \arg\gamma_1'(a)$$

and (36) follows.

12.24. Angles between curves on Riemann surfaces. One can define angles between paths on a Riemann surface X similarly as in Subsection 12.23 by using local charts. Since transition maps between charts are conformal maps and hence angle-preserving, this leads to a well-defined notion.

In particular, if γ_1 and γ_2 are suitable path in $\widehat{\mathbb{C}}$ with initial point ∞, then

$$\angle_\infty(\gamma_1, \gamma_2) := \angle_0(I \circ \gamma_1, I \circ \gamma_2),$$

where $I\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, $I(z) = 1/z$ is the inversion.

12.25. Möbius transformations preserve angles between paths. Let $S \in$ Möb, and γ_1, γ_2 be smooth paths in $\widehat{\mathbb{C}}$ with initial point $p \in \widehat{\mathbb{C}}$. Let $q = S(p)$. Then

$$\angle_p(\gamma_1, \gamma_2) = \angle_q(S \circ \gamma_1, S \circ \gamma_2).$$

To see this, assume that

$$S(z) = \frac{az + b}{cz + d} \quad \text{with } a, b, c, d \in \mathbb{C},\ ad - bc \neq 0.$$

We consider several cases similar to the cases in Example 11.12.

Case 1: $p, q \in \mathbb{C}$. By the considerations in 12.23 it suffices to show that $S'(p) \neq 0$. This is true, since S is a biholomorphism, or more explicitly,

$$S'(p) = \frac{a(cp + d) - c(ap + b)}{(cp + d)^2} = \frac{ad - bc}{(cp + d)^2} \neq 0.$$

Note that $cp + d \neq 0$, because $q = S(p) \in \mathbb{C}$.

Case 2: $p = \infty$, $q \in \mathbb{C}$. Then we consider $z \mapsto S(1/z) =: T(z)$, and have to check whether $T'(0) \neq 0$. Since $T \in$ Möb, and $T(0) = S(\infty) = q \in \mathbb{C}$, this follows from the considerations in Case 1.

There are two more cases, namely where $p \in \mathbb{C}$, $q = \infty$, and where $p = q = \infty$. They can be treated in a similar way as in Case 2.

Example 12.26. Let

$$S(z) = \frac{1-z}{1+z}.$$

What is the set $S(\mathbb{D})$?

Solution: S has real coefficients; so $S(\widehat{\mathbb{R}}) = \widehat{\mathbb{R}}$ (see 12.14). We have $-1, 1 \in \partial\mathbb{D}$, and $S(-1) = \infty$ and $S(1) = 0$; so $S(\partial\mathbb{D})$ is a line L. Since $\partial\mathbb{D}$ meets $\widehat{\mathbb{R}}$ orthogonally at 1, this line L meets $S(\widehat{\mathbb{R}}) = \widehat{\mathbb{R}}$ orthogonally at $S(1) = 0$ (see 12.25). It follows that L is the imaginary axis. Since $S(0) = 1$, we conclude that $S(\mathbb{D}) = \{w \in \mathbb{C} : \operatorname{Im} w > 0\}$ is the open right half-plane.

13 Schwarz's Lemma

Theorem 13.1 (Schwarz's Lemma)**.** *Let* $f\colon \mathbb{D} \to \mathbb{C}$ *be a holomorphic function satisfying* $f(0) = 0$ *and* $|f(z)| \le 1$ *for* $z \in \mathbb{D}$*. Then*

$$|f(z)| \le |z| \quad \text{for all } z \in \mathbb{D}, \tag{37}$$

and

$$|f'(0)| \le 1. \tag{38}$$

If there is a point $z \in \mathbb{D} \setminus \{0\}$ *for which we have equality in* (37)*, or if we have equality in* (38)*, then there exists* $\theta \in \mathbb{R}$ *such that*

$$f(z) = e^{i\theta} z \quad \text{for all } z \in \mathbb{D}.$$

Proof. Since f has a zero at 0, we can write this function in the form $f(z) = zh(z)$ for $z \in \mathbb{D}$, where $h\colon \mathbb{D} \to \mathbb{C}$ is holomorphic (cf. Theorem 8.3).

Claim: $|h(z)| \le 1$ for all $z \in \mathbb{D}$.

To see this, let $z_0 \in \mathbb{D}$, and consider an arbitrary number $r > 0$ with $|z_0| < r < 1$. Then h is continuous on $\overline{B}(0, r) \subseteq \mathbb{D}$ and holomorphic on $B(0, r)$. Be the Maximum Modulus Principle (cf. Theorem 8.9), we have

$$|h(z_0)| \le \max_{z \in \partial B(0,r)} |h(z)| = \max_{z \in \partial B(0,r)} \frac{|f(z)|}{|z|} \le \frac{1}{r}.$$

Letting $r \to 1$, we conclude $|h(z_0)| \le 1$, and the claim follows.

The claim implies that

$$|f(z)| = |zh(z)| \le |z|$$

for $z \in \mathbb{D}$, and

$$|f'(0)| = |h(0)| \le 1.$$

So we have established (37) and (38).

If we have equality in (37) for some $z = z_0 \in \mathbb{D} \setminus \{0\}$, then $|h(z_0)| = 1$, and so h must be constant by the Maximum Modulus Principle (cf. Theorem 8.6). This is also true if $|f'(0)| = 1 = |h(0)|$.

In both cases, $h \equiv e^{i\theta}$ for some $\theta \in \mathbb{R}$, and so $f(z) = e^{i\theta} z$ for $z \in \mathbb{D}$. □

13.2. The automorphisms of a region. Let $\Omega \subseteq \widehat{\mathbb{C}}$ be a region. A conformal map from Ω onto Ω is called a *(conformal) automorphism* of Ω. The set of all automorphisms of Ω is denoted by $\mathrm{Aut}(\Omega)$. This is a group under compositions of maps, the so-called *automorphism group* of Ω.

Theorem 13.3 (Automorphisms of the unit disk). *A map $f\colon \mathbb{D} \to \mathbb{C}$ is a conformal automorphism of $\mathbb{D}$ if and only if f is a Möbius transformation of the form*

$$f(z) = e^{i\theta}\frac{z-a}{1-\bar{a}z}, \tag{39}$$

where $\theta \in \mathbb{R}$, $a \in \mathbb{D}$.

By this theorem any point $z_0 \in \mathbb{D}$ can be send to 0 by a suitable Möbius transformation $f \in \mathrm{Aut}(\mathbb{D})$ (take $a = z_0$ in (39)). The rotation factor $e^{i\theta}$ can be used to send a second point $w_0 \in \mathbb{D}$ to the positive real axis, and so into $[0,1)$. In other words, if $z_0, w_0 \in \mathbb{D}$, then there exists $f \in \mathrm{Aut}(\mathbb{D})$ such that $f(z_0) = 0$ and $f(w_0) \in [0,1)$.

Proof of Theorem 13.3. $\Leftarrow$: Every $f \in$ Möb of the form (39) is an automorphism of $\mathbb{D}$; indeed,

$$|f(e^{it})| = \left| e^{i\theta}\frac{e^{it}-a}{1-\bar{a}e^{it}} \right| = \frac{|1-ae^{-it}|}{|1-ae^{-it}|} = 1$$

for $t \in \mathbb{R}$. Hence $f(\partial\mathbb{D}) = \partial\mathbb{D}$. Since $f(0) = -e^{i\theta}a \in \mathbb{D}$, we have $f(\mathbb{D}) = \mathbb{D}$. So f maps $\mathbb{D}$ onto $\mathbb{D}$ bijectively. Since f and the inverse of f are Möbius transformations, and hence holomorphic, it follows that f is an automorphism of $\mathbb{D}$.

$\Rightarrow$: Let $f \in \mathrm{Aut}(\mathbb{D})$ be arbitrary. Then there exists $a \in \mathbb{D}$ such that $f(a) = 0$. Define

$$g(z) = \frac{z+a}{1+\bar{a}z}.$$

Then $g \in \mathrm{Aut}(\mathbb{D})$ by the first part of the proof. Let $\varphi = f \circ g$. Then $\varphi \in \mathrm{Aut}(\mathbb{D})$, and

$$\varphi(0) = f(g(0)) = f(a) = 0.$$

We also have $\psi := \varphi^{-1} \in \mathrm{Aut}(\mathbb{D})$, and $\psi(0) = 0$. Schwarz's Lemma gives

$$|\varphi'(0)| \leq 1 \quad \text{and} \quad |\psi'(0)| \leq 1.$$

On the other hand, $\psi(\varphi(z)) = z$ for $z \in \mathbb{D}$, and so

$$|\psi'(0)| \cdot |\varphi'(0)| = 1.$$

It follows that

$$|\psi'(0)| = 1 = |\varphi'(0)|,$$

and so by the second part Schwarz's Lemma there exists $\theta \in \mathbb{R}$ such that

$$\varphi(z) = f(h(z)) = e^{i\theta} z$$

for $z \in \mathbb{D}$. Computation gives

$$g^{-1}(z) = \frac{z-a}{1-\bar{a}z},$$

which implies that

$$f(z) = f(g(g^{-1}(z))) = e^{i\theta} g^{-1}(z) = e^{i\theta} \frac{z-a}{1-\bar{a}z}.$$

This shows that f can be written as in (39). $\square$

13.4. Hyperbolic geometry. Based on Theorem 13.3 one can show that

$$\frac{2|S'(z)|}{1-|S(z)|^2} = \frac{2}{1-|z|^2} \tag{40}$$

for all $S \in \mathrm{Aut}(\mathbb{D})$ and $z \in \mathbb{D}$ (exercise!).

If $\gamma\colon [a,b] \to \mathbb{D}$ is a piecewise smooth path, one defines its *hyperbolic length* as

$$\ell_h(\gamma) = \int_\gamma \frac{2|dz|}{1-|z|^2} := \int_a^b \frac{2|\gamma'(t)|}{1-|\gamma(t)|^2}\, dt.$$

Then it follows from (40) that

$$\ell_h(S \circ \gamma) = \ell_h(\gamma) \tag{41}$$

for all $S \in \mathrm{Aut}(\mathbb{D})$, i.e., the hyperbolic length of a path is invariant under automorphisms of $\mathbb{D}$.

The *hyperbolic metric* on $\mathbb{D}$ is defined by

$$d_h(z,w) := \inf_\gamma \ell_h(\gamma) \quad \text{for } z, w \in \mathbb{D}, \tag{42}$$

where the infimum is taken over all piecewise smooth paths in $\mathbb{D}$ with endpoints z and w. By (41) each $S \in \mathrm{Aut}(\mathbb{D})$ is an isometry of $\mathbb{D}$ equipped with hyperbolic metric.

If $z = 0$ and $w = u \in \mathbb{D}$, then one can show that the (parametrized) segment $[0,u]$ is the unique *hyperbolic geodesic segment*, i.e, the unique (up to reparametrization) piecewise smooth path γ for which the infimum in (42) is attained. Hence

$$\begin{aligned} d_h(0,u) &= \int_0^{|u|} \frac{2dt}{1-t^2} = \int_0^{|u|} \left(\frac{1}{1+t} + \frac{1}{1-t}\right) dt \\ &= (\log(1+t) - \log(1-t))\Big|_0^{|u|} \\ &= \log\left(\frac{1+|u|}{1-|u|}\right) = 2\,\mathrm{artanh}\,|u|. \end{aligned} \tag{43}$$

Using this, the fact that every $S \in \mathrm{Aut}(\mathbb{D})$ is a hyperbolic isometry, and the remark after Theorem 13.3, one easily obtains the explicit expression

$$d_h(z,w) = \log\left(\frac{1+\left|\frac{z-w}{1-\bar{z}w}\right|}{1-\left|\frac{z-w}{1-\bar{z}w}\right|}\right) = 2\,\mathrm{artanh}\left|\frac{z-w}{1-\bar{z}w}\right| \tag{44}$$

for the hyperbolic distance between two points $z, w \in \mathbb{D}$.

The hyperbolic geodesic segment Σ joining two points $z, w \in \mathbb{D}$ is the image of a an interval of the form $[0, r]$, $0 \le r < 1$, under a suitable map $S \in \mathrm{Aut}(\mathbb{D})$. Since $\widehat{\mathbb{R}} \supseteq [0, r]$ meets the unit circle orthogonally and S preserves angles, Σ is contained in a unique circle C that passes through z and w, and meets $\partial\mathbb{D}$ orthogonally. Then Σ is given by the subarc of C in $\mathbb{D}$ with endpoints z and w.

The unit circle $\mathbb{D}$ equipped with the hyperbolic metric is a model of the hyperbolic plane in non-Euclidean geometry, called the *Poincaré model* of the hyperbolic plane.

Theorem 13.5 (Schwarz-Pick Lemma)**.** *Let $f\colon \mathbb{D} \to \mathbb{C}$ be a holomorphic function with $f(\mathbb{D}) \subseteq \mathbb{D}$. Then we have*

$$\left|\frac{f(z_1) - f(z_2)}{1 - \overline{f(z_1)}f(z_2)}\right| \le \left|\frac{z_1 - z_2}{1 - \overline{z_1}z_2}\right| \tag{45}$$

for all $z_1, z_2 \in \mathbb{D}$, and

$$\frac{|f'(z)|}{1 - |f(z)|^2} \le \frac{1}{1 - |z|^2} \tag{46}$$

for all $z \in \mathbb{D}$.

If we have equality in (45) *for some points $z_1, z_2 \in \mathbb{D}$, $z_1 \neq z_2$, or if we have equality in* (46) *for some $z \in \mathbb{D}$, then $f \in \mathrm{Aut}(\mathbb{D})$.*

Remark 13.6. If we define

$$B(z_1, z_2) := \left| \frac{z_1 - z_2}{1 - \overline{z_1} z_2} \right|,$$

then (cf. (44)),

$$d_h(z_1, z_2) = \log\left(\frac{1 + B(z_1, z_2)}{1 - B(z_1, z_2)}\right).$$

Since the function $r \mapsto \log\left(\dfrac{1+r}{1-r}\right)$ is strictly increasing on $[0,1)$, inequality (45) is equivalent to

$$d_h(f(z_1), f(z_2)) \leq d_h(z_1, z_2).$$

So a holomorphic map $f\colon \mathbb{D} \to \mathbb{D}$ does not increase hyperbolic distances of points.

If $f \in \mathrm{Aut}(\mathbb{D})$, then f is a hyperbolic isometry and we have equality in (45) for *all* $z_1, z_2 \in \mathbb{D}$, and also in (46) for *all* $z \in \mathbb{D}$ by (40).

Proof of Theorem 13.5 (Outline). The idea is to reduce the statements to corresponding statements in Schwarz's Lemma.

Let $z_1, z_2 \in \mathbb{D}$ be arbitrary, and define $w_1 = f(z_1)$, $w_2 = f(z_2)$,

$$S(z) = \frac{z - z_1}{1 - \overline{z_1} z} \quad \text{and} \quad T(w) = \frac{w - w_1}{1 - \overline{w_1} w}.$$

Then $S, T \in \mathrm{Aut}(\mathbb{D})$, $S(z_1) = 0$, and $T(w_1) = 0$.

Define $g = T \circ f \circ S^{-1}$. Then $g \in H(\mathbb{D})$, $g(0) = 0$, and $g(\mathbb{D}) \subseteq \mathbb{D}$. So Schwarz's Lemma gives

$$|g(u)| \leq |u| \tag{47}$$

for all $u \in \mathbb{D}$, in particular for $u = S(z_2)$. This translates into (45). Similarly, we have

$$|g'(0)| \leq 1, \tag{48}$$

which after some computation gives (48) (for $z = z_1$).

In case of (non-trivial) equality in (45), or of equality in (46), we have equality in (47) or (48). Then $g(z) \equiv e^{i\theta} z$ for some $\theta \in \mathbb{R}$, and so $g \in \mathrm{Aut}(\mathbb{D})$. Then $f = T^{-1} \circ g \circ S \in \mathrm{Aut}(\mathbb{D})$. □

13.7. Conformal metrics. Let $U \subseteq \mathbb{C}$ be a region, and $\rho\colon U \to [0, \infty)$ be a continuous function such that $\rho^{-1}(0)$ consists of isolated points in U (the "singularities" of ρ). Given such a *conformal density*, one defines the *ρ-length* of a piecewise smooth path $\gamma\colon [a, b] \to U$ by

$$\ell_\rho(\gamma) = \int_\gamma \rho(z)\,|dz| := \int_a^b \rho(\gamma(t))|\gamma'(t)|\,dt,$$

and an associated *conformal metric* given by

$$d_\rho(z, w) = \inf_\gamma \ell_\rho(\gamma) \quad \text{for } z, w \in U,$$

where the infimum is taken over all piecewise smooth paths γ in U joining z and w.

Then d_ρ is a metric on U that induces the standard topology (to prove this one needs that $\rho^{-1}(0)$ consists of isolated points in U). One says that this metric is given by the (symbolic) *length element*

$$ds = \rho(z)\,|dz|.$$

For example, the hyperbolic metric on $\mathbb{D}$ is induced by the length element

$$ds = \rho_h(z)\,|dz|,$$

where

$$\rho_h(z) := \frac{2}{1 - |z|^2}, \quad z \in \mathbb{D},$$

is the *hyperbolic density.*

13.8. Pull-backs of conformal metrics. Let $U, V \subseteq \mathbb{C}$ be regions, $f\colon U \to V$ be a non-constant holomorphic map, and $ds = \rho(w)\,|dw|$ be a conformal metric on V. Here we use w to denote points in V, and z for points in U. Then the *pull-back* $f^*(ds)$ of ds by the map f (given by $z \mapsto w = f(z)$) is the conformal metric on U with length element

$$\begin{aligned} f^*(ds) &= \rho(w)\,|dw| \\ &= \rho(f(z))\left|\frac{dw}{dz}\right|\,|dz| \\ &= \rho(f(z))|f'(z)|\,|dz|. \end{aligned}$$

(Of course this "computation" is purely symbolic and is intended to motivate the definition of the length element in the last line). Note that this pull-back operation is functorial in the sense that

$$(g \circ f)^*(ds) = f^*(g^*(ds))$$

for compositions of holomorphic maps (this immediately follows from the chain rule).

If the map f is conformal, then f is an isometry between V equipped with the conformal metric given by ds and U equipped with $f^*(ds)$.

If we have two conformal metrics

$$ds_1 = \rho_1(z)\,|dz| \quad \text{and} \quad ds_2 = \rho_2(z)\,|dz|$$

on U, we write $ds_1 \leq ds_2$ iff $\rho_1 \leq \rho_2$. Then the Schwarz-Pick Lemma is equivalent to the statement

$$f^*(ds_h) \leq ds_h, \tag{49}$$

where $ds_f = \rho_h(z)\,|dz|$ is the hyperbolic metric on $\mathbb{D}$, and $f\colon \mathbb{D} \to \mathbb{D}$ is a holomorphic map; indeed, (49) is the same as (46), from which on can derive (47) by integration.

Example 13.9. (a) Let

$$w = f(z) = \frac{1+iz}{1-iz}.$$

Then f is a Möbius transformation that maps the upper half-plane $H = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ onto $\mathbb{D}$. If we pull-back the hyperbolic metric $ds_h = \rho_h(w)\,|dw|$ on $\mathbb{D}$ by f to H, then we get

$$f^*(ds_h) = \frac{|dz|}{\operatorname{Im} z}.$$

This conformal metric on H gives an isometric model of the hyperbolic plane, called the *upper half-plane model*.

(b) One can represent the unit sphere in $\mathbb{R}^3$ equipped with the spherical metric d_s (given by geodesic length between points) as a conformal metric on $\widehat{\mathbb{C}}$, namely by

$$ds = \frac{2|dz|}{1+|z|^2}$$

(suitably interpreted at $z = \infty$). One can show that that then

$$d_s(z, w) = 2 \arctan \left| \frac{z - w}{1 + \bar{z} w} \right|$$

for $z, w \in \widehat{\mathbb{C}}$. Note that this expression is very similar to (44).

14 Winding numbers

Remark 14.1. We want to define an integer that counts how often a loop $\gamma\colon [0,1] \to \mathbb{C}$ winds around a given point $z \in \mathbb{C}$, say $z = 0$. This only makes sense if $0 \notin \gamma^*$. The idea is to look at the total change of the polar angle if we move along the loop γ. For this we want to write γ in the form $\gamma(t) = e^{\alpha(t)}$, $t \in [0,1]$, where $\alpha\colon [0,1] \to \mathbb{C}$ is continuous. Then the imaginary part of α corresponds to the polar angle of points on γ, and we define the winding number of γ around 0 as

$$\text{winding number} = \frac{1}{2\pi}(\operatorname{Im}\alpha(1) - \operatorname{Im}\alpha(0)). \tag{50}$$

The first problem that one has to address here is the existence of α. We will study is problem in greater generality based on the following definition.

Let Z be a metric space, and $f\colon Z \to \mathbb{C}^*$ be a continuous map. A continuous map $\widetilde{f}\colon Z \to \mathbb{C}$ is called a *lift* of f (under the exponential map) if $f = \exp \circ \widetilde{f}$, i.e., if the diagram

$$\begin{tikzcd} & \mathbb{C} \arrow[d, "\exp"] \\ Z \arrow[ur, "\widetilde{f}"] \arrow[r, "f"] & \mathbb{C}^* \end{tikzcd}$$

commutes. So finding a continuous function α as above is the same as finding a lift of the loop γ.

Remark 14.2. The map $\exp\colon \mathbb{C} \to \mathbb{C}^*$ is an example of a covering map. By definition a continuous map $p\colon X \to Y$ between topological spaces X and Y is a *covering map* if each point $y \in Y$ has an open neighborhood V with the following property: there exists an index set I such that $p^{-1}(V)$ can be written as a union

$$p^{-1}(V) = \bigcup_{i \in I} U_i$$

of pairwise disjoint open sets $U_i \subseteq X$ such that $p|U_i$ is a homeomorphism from U_i onto V for each $i \in I$. Such a neighborhood V is called a *fundamental neighborhood* or an *evenly covered neighborhood* of the point y.

To see that $\exp\colon \mathbb{C} \to \mathbb{C}^*$ is indeed a covering map, let $w_0 \in \mathbb{C}^*$ be arbitrary. Then $w_0 = re^{i\alpha}$ for some $r > 0$ and $\alpha \in \mathbb{R}$. If we define

$$V = \{\rho e^{it} : \rho > 0,\ \alpha - \pi < t < \alpha + \pi\} = \mathbb{C} \setminus \{\rho e^{i(\alpha+\pi)} : \rho \geq 0\},$$

then V is an evenly covered neighborhood of w_0. Indeed, V is an open set containing w_0, and so a neighborhood of w_0. Moreover,

$$\exp^{-1}(V) = \bigcup_{k \in \mathbb{Z}} U_k,$$

where

$$U_k = \{z \in \mathbb{C} : \alpha + (2k-1)\pi < \operatorname{Im} z < \alpha + (2k+1)\pi\}$$

for $k \in \mathbb{Z}$. The sets U_k, $k \in \mathbb{Z}$, are open and pairwise disjoint, and for each $k \in \mathbb{Z}$ the restriction $\exp|U_k \colon U_k \to V$ is a holomorphic bijection of U_k onto V. Then $\exp|U_k$ has a holomorphic, and in particular continuous inverse which shows that $\exp|U_k$ is a homeomorphism of U_k onto V.

Note that $V' = B(w, |w_0|) \subseteq V$ is also an evenly covered neighborhood of w_0.

Proposition 14.3. *Let Z be a compact and connected metric space, and $f \colon Z \to \mathbb{C}^*$ be a continuous map.*

(a) *Suppose that $\widetilde{f}_1$ and $\widetilde{f}_2$ are lifts of f and that there exists a point $z_0 \in Z$ such that $\widetilde{f}_1(z_0) = \widetilde{f}_2(z_0)$. Then $\widetilde{f}_1 = \widetilde{f}_2$.*

(b) *Let $z_0 \in Z$, define $w_0 = f(z_0)$, and suppose that*

$$\operatorname{diam}(f(Z)) < \operatorname{dist}(0, f(Z)).$$

If $u_0 \in \exp^{-1}(w_0)$, then there exists a unique lift $\widetilde{f}$ of f with $\widetilde{f}(z_0) = u_0$.

The first statement says that under the given assumptions, lifts are uniquely determined by the image of one point.

Note that in (b) we have $\operatorname{dist}(0, f(Z)) > 0$, because $f(Z)$ is a compact subset of $\mathbb{C}^*$. The statement in (b) essentially says that lifts exist if the image $f(Z)$ is sufficiently small (depending on $\operatorname{dist}(0, f(Z))$).

Proof. (a) We have

$$\exp(\widetilde{f}_1 - \widetilde{f}_2) = \exp(\widetilde{f}_1)/\exp(\widetilde{f}_2) = f/f = 1.$$

Since Z is connected, we conclude $\widetilde{f}_1 - \widetilde{f}_2 \equiv$ const. (see Subsection 10.2 for a similar reasoning). By assumption $\widetilde{f}_1(z_0) - \widetilde{f}_2(z_0) = 0$, and so $\widetilde{f}_1 = \widetilde{f}_2$.

(b) Since $u_0 \in \exp^{-1}(w_0)$ and $V = B(w_0, |w_0|)$ is an evenly covered neighborhood of w_0 for the covering map $\exp\colon \mathbb{C} \to \mathbb{C}^*$, there exists a local inverse of exp on V, i.e., a branch L of the logarithm on V such that $L(w_0) = u_0$. Note that for each $z \in Z$ we have

$$\begin{aligned}|f(z) - w_0| = |f(z) - f(z_0)| &\leq \operatorname{diam}(f(Z)) \\ &< \operatorname{dist}(0, f(Z)) \leq |f(z_0)| = |w_0|.\end{aligned}$$

Hence $f(Z) \subseteq B(w_0, |w_0|)$. So $\widetilde{f} := L \circ f$ is defined. It is immediate that $\widetilde{f}$ is a lift of f with $\widetilde{f}(z_0) = u_0$. Moreover, by (a) this lift is unique. □

Lemma 14.4. *Let X and Y be metric spaces, and $f\colon X \to Y$ be a map. Suppose that there are finitely many closed sets $F_1, \dots, F_n \subseteq X$ such that $f|F_k$ is continuous for each $k \in \{1, \dots, n\}$ and $X = \bigcup_{k=1}^{n} F_k$. Then f is continuous.*

Proof. From the assumptions it easily follows that if $A \subseteq Y$ is closed, then $f^{-1}(A)$ is also closed. Hence f is continuous (see Proposition 2.22). □

Theorem 14.5 (Existence of lifts)**.** (a) *Let $\gamma\colon [0,1] \to \mathbb{C}^*$ be a continuous path, and $u_0 \in \exp^{-1}(\gamma(0))$. Then there exists a unique lift $\widetilde{\gamma}\colon [0,1] \to \mathbb{C}$ of γ with $\widetilde{\gamma}(0) = u_0$. If γ is piecewise smooth, then $\widetilde{\gamma}$ is also piecewise smooth.*

(b) *Let $H\colon [0,1] \times [0,1] \to \mathbb{C}^*$ be continuous and $u_0 \in \exp^{-1}(H(0,0))$. Then there exists a unique lift $\widetilde{H}\colon [0,1] \times [0,1] \to \mathbb{C}$ of H such that $\widetilde{H}(0,0) = u_0$.*

The first part of (a) is true in greater generality for covering maps and called the "path lifting property" of such maps. The statement in (b) can be considered as a special case of the "Homotopy Lifting Theorem" for covering maps.

Proof (Outline). (a) One can find a fine partition of $[0,1]$ given by points $t_0 = 0 < t_1 < \cdots < t_n = 1$ such that each of the paths $\gamma_k := \gamma|[t_{k-1}, t_k]$ has small enough image so that we can apply Proposition 14.3 (b). Then we successively lift the paths $\gamma_1, \dots, \gamma_n$ to paths $\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_n$. We can do this in such a way that $\widetilde{\gamma}_1(0) = u_0$, and so that the endpoint $\widetilde{\gamma}_k(t_k)$ of a lift $\widetilde{\gamma}_k$ coincides with the initial point $\widetilde{\gamma}_{k+1}(t_k)$ of the next lift $\widetilde{\gamma}_{k+1}$.

By Lemma 14.4 these lifts $\widetilde{\gamma}_k$ paste together to a continuous map $\widetilde{\gamma}\colon [0,1] \to \mathbb{C}$ such that $\gamma|[t_{k-1}, t_k] = \widetilde{\gamma}_k$. Then $\exp \circ \widetilde{\gamma} = \gamma$ and $\widetilde{\gamma}(0) = \widetilde{\gamma}_1(0) = u_0$, and so $\widetilde{\gamma}$ is a lift of γ with the desired properties. It is unique as follows from Proposition 14.3 (a).

If γ is piecewise smooth, then we can choose the partition so that the paths γ_k are (C^1-)smooth. Since each lift $\widetilde{\gamma}_k$ can be obtained by composing a branch of the logarithm with γ_k, the lift $\widetilde{\gamma}_k$ is also smooth. Hence $\widetilde{\gamma}$ is piecewise smooth.

(b) The basic idea for the proof is the same as in (a). We subdivide $[0,1] \times [0,1]$ into small squares Q so that we can apply Proposition 14.3 (b) to each map $H|Q$. We can ensure that $(0,0)$ is mapped to u_0 by a suitable lift. Moreover, based on Proposition 14.3 (a) one can obtain lifts of the maps $H|Q$ so they match along common boundary points of squares Q. Again by Lemma 14.4 these lifts will paste together to a lift $\widetilde{H}$ of H with $\widetilde{H}(0,0) = u_0$. Uniqueness of $\widetilde{H}$ follows from Proposition 14.3 (a). The details are left as an excercise. □

Definition 14.6 (Index of a loop with respect to a point)**.** Let $\gamma\colon [0,1] \to \mathbb{C}$ be a loop with $0 \notin \gamma^*$. Then we define the *index of* γ *with respect to* 0 ("the winding number of γ around 0"), denoted by $\operatorname{ind}_\gamma(0)$, as follows: Pick a lift $\widetilde{\gamma}\colon [0,1] \to \mathbb{C}$ of γ. Then

$$\operatorname{ind}_\gamma(0) := \frac{1}{2\pi i}(\widetilde{\gamma}(1) - \widetilde{\gamma}(0)). \tag{51}$$

More generally, the *index of* γ *with respect to a point* $w \in \mathbb{C} \setminus \gamma^*$ ("the winding number of γ around w"), is defined as

$$\operatorname{ind}_\gamma(w) := \operatorname{ind}_{\gamma - w}(0).$$

Remark 14.7. (a) Note that a lift $\widetilde{\gamma}$ as in the previous definition exists by Theorem 14.5 (a). Moreover, $\operatorname{ind}_\gamma(0)$ is well-defined. Indeed, if $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ are two lifts of γ, then

$$\exp(\widetilde{\gamma}_1 - \widetilde{\gamma}_2) = \exp(\widetilde{\gamma}_1)/\exp(\widetilde{\gamma}_2) = \gamma/\gamma = 1,$$

and so $\widetilde{\gamma}_1 - \widetilde{\gamma}_2 = \text{const.}$ Hence $\widetilde{\gamma}_1 = \widetilde{\gamma}_2 + \text{const.}$, which implies

$$\widetilde{\gamma}_1(1) - \widetilde{\gamma}_1(0) = \widetilde{\gamma}_2(1) - \widetilde{\gamma}_2(0).$$

(b) The index is an integer. Indeed, since γ is a loop, we have

$$\exp(\widetilde{\gamma}(0)) = \gamma(0) = \gamma(1) = \exp(\widetilde{\gamma}(1)).$$

Hence $\widetilde{\gamma}(1) = \widetilde{\gamma}(0) + 2\pi i k$ for some $k \in \mathbb{Z}$, and

$$\operatorname{ind}_\gamma(0) = \frac{1}{2\pi i}(\widetilde{\gamma}(1) - \widetilde{\gamma}(0)) = k \in \mathbb{Z}.$$

(c) The definition given in (51) agrees with a similar definition based on (50) with $\alpha = \widetilde{\gamma}$. To see this note that

$$\operatorname{Re}(\widetilde{\gamma}(1)) = \log|\gamma(1)| = \log|\gamma(0)| = \operatorname{Re}(\widetilde{\gamma}(0)).$$

(d) In the previous discussion the assumption that the loops are defined on $[0,1]$ was of course just for convenience. One defines winding numbers for loops defined of arbitrary compact subintervals of $\mathbb{R}$ in essentially the same way as for loops on $[0,1]$.

Theorem 14.8. *Let $\gamma\colon [0,1] \to \mathbb{C}$ be a piecewise smooth loop and $w \in \mathbb{C}\setminus\gamma^*$. Then*

$$\operatorname{ind}_\gamma(w) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-w}.$$

Proof. By Theorem 14.5 (a) the path $\gamma - w$ has a piecewise smooth lift $\widetilde{\gamma}$. Then

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-w} &= \frac{1}{2\pi i}\int_0^1 \frac{\gamma'(t)}{\exp(\widetilde{\gamma}(t))}\,dt = \frac{1}{2\pi i}\int_0^1 \widetilde{\gamma}'(t)\,dt \\ &= \frac{1}{2\pi i}(\widetilde{\gamma}(1) - \widetilde{\gamma}(0)) = \operatorname{ind}_\gamma(0).\end{aligned}$$

□

Proposition 14.9. *Let $\alpha, \gamma_1, \gamma_2\colon [0,1] \to \mathbb{C}$ be loops, and assume $\gamma_1(1) = \gamma_2(0)$.*

(a) *Define $\beta\colon [0,1] \to \mathbb{C}$ by $\beta(t) = \alpha(1-t)$ for $t \in [0,1]$ (so β is the path opposite to α). If $z \in \mathbb{C}\setminus\alpha^* = \mathbb{C}\setminus\beta^*$, then*

$$\operatorname{ind}_\beta(z) = -\operatorname{ind}_\alpha(z).$$

(b) *Define* $\gamma\colon [0,2] \to \mathbb{C}$ *by*

$$\gamma(t) = \begin{cases} \gamma_1(t) & \textit{for } t \in [0,1], \\ \gamma_2(t-1) & \textit{for } t \in [1,2]. \end{cases}$$

If $z \in \mathbb{C}$ *and* $z \notin \gamma^* = \gamma_1^* \cup \gamma_2^*$, *then*

$$\operatorname{ind}_\gamma(z) = \operatorname{ind}_{\gamma_1}(z) + \operatorname{ind}_{\gamma_2}(z).$$

Proof. Without loss of generality $z = 0$.

(a) Let $\widetilde{\alpha}\colon [0,1] \to \mathbb{C}$ be a lift of α. Then $\widetilde{\beta}\colon [0,1] \to \mathbb{C}$, $\widetilde{\beta}(t) := \widetilde{\alpha}(1-t)$, $t \in [0,1]$, is a lift of $\widetilde{\beta}$, and so

$$\operatorname{ind}_\beta(0) = \frac{1}{2\pi i}(\widetilde{\beta}(1) - \widetilde{\beta}(0)) = -\frac{1}{2\pi i}(\widetilde{\alpha}(1) - \widetilde{\alpha}(0)) = -\operatorname{ind}_\alpha(z).$$

(b) First note that γ is a loop, because

$$\gamma(0) = \gamma_1(0) = \gamma_1(1) = \gamma_2(0) = \gamma_2(1) = \gamma(2).$$

Let $\widetilde{\gamma}_1$ be a lift of γ_1. Then

$$\widetilde{\gamma}_1(1) \in \exp^{-1}(\gamma_1(1)) = \exp^{-1}(\gamma_2(0)),$$

and so there exists a lift $\widetilde{\gamma}_2\colon [0,1] \to \mathbb{C}$ of γ_2 such that $\widetilde{\gamma}_2(0) = \widetilde{\gamma}_1(1)$. Define

$$\widetilde{\gamma}(t) = \begin{cases} \widetilde{\gamma}_1(t) & \text{for } t \in [0,1], \\ \widetilde{\gamma}_2(t-1) & \text{for } t \in [1,2]. \end{cases}$$

Then $\widetilde{\gamma}$ is a lift of γ, and so

$$\begin{aligned} \operatorname{ind}_\gamma(0) &= \frac{1}{2\pi i}(\widetilde{\gamma}(2) - \widetilde{\gamma}(0)) = \frac{1}{2\pi i}(\widetilde{\gamma}(2) - \widetilde{\gamma}(1) + \widetilde{\gamma}(1) - \widetilde{\gamma}(0)) \\ &= \frac{1}{2\pi i}(\widetilde{\gamma}_2(1) - \widetilde{\gamma}_2(0)) + \frac{1}{2\pi i}(\widetilde{\gamma}_1(1) - \widetilde{\gamma}_1(0)) \\ &= \operatorname{ind}_{\gamma_1}(0) + \operatorname{ind}_{\gamma_2}(0). \end{aligned}$$

$\square$

Definition 14.10. (a) Let $U \subseteq \mathbb{C}$ and $\gamma_0, \gamma_1 \colon [0,1] \to U$ be loops in U. We say that γ_0 and γ_1 are *(loop-)homotopic in* U, written $\gamma_0 \sim \gamma_1$ in U, if there exists a continuous map $H\colon [0,1]\times[0,1] \to \mathbb{C}$ such that $H(\cdot,0) = \gamma_0$, $H(\cdot,1) = \gamma_1$, and such that $H_t := H(\cdot,t)$ is a loop for each $t \in [0,1]$ (i.e., $H(0,t) = H(1,t)$ for all $t \in [0,1]$).

(b) We say that a region $\Omega \subseteq \mathbb{C}$ is *simply connected* if every loop γ in Ω is *null-homotopic in* Ω, i.e., γ is loop-homotopic in Ω to a constant path (written $\gamma \sim 0$ in Ω).

A map H as in (a) is called a *loop-homotopy* or simply a *homotopy* between the loops γ_0 and γ_1.

Remark 14.11. (a) If $\Omega \subseteq \mathbb{C}$ is a convex region, then it is simply connected. To see this, let $\gamma_0\colon [0,1] \to \Omega$ be a loop in Ω. Pick a point $a \in \Omega$, and define

$$H(s,t) := (1-t)\gamma_0(s) + ta$$

for $s,t \in [0,1]\times[0,1]$. Then H is a loop-homotopy in Ω between γ_0 and the constant path $s \mapsto \gamma_1(s) := a$.

(b) One can show that a region $\Omega \subseteq \mathbb{C}$ is simply connected if and only if $\widehat{\mathbb{C}} \setminus \Omega$ is connected. The proof is surprisingly difficult and will be given later.

Theorem 14.12. *Let $\gamma_0, \gamma_1 \colon [0,1] \to \mathbb{C}$ be paths and $z \in \mathbb{C} \setminus (\gamma_0^* \cup \gamma_1^*)$. Then γ_0 and γ_1 are homotopic in $\mathbb{C}\setminus\{z\}$ if and only if*

$$\operatorname{ind}_{\gamma_1}(z) = \operatorname{ind}_{\gamma_2}(z).$$

Proof. Without loss of generality we may assume that $z = 0$.

$\Rightarrow$: Suppose that there exists a homotopy $H\colon [0,1]\times[0,1] \to \mathbb{C}^*$ between γ_0 and γ_1 in $\mathbb{C}^* = \mathbb{C}\setminus\{z\}$. Then $H(\cdot,0) = \gamma_0$, $H(\cdot,1) = \gamma_1$, and $\gamma_t := H(\cdot,t)$ is a loop for each $t \in [0,1]$. By Theorem 14.5 (b) there exists a lift $\widetilde{H}\colon [0,1]\times[0,1] \to \mathbb{C}$ of H. Then $\widetilde{\gamma}_t := \widetilde{H}(\cdot,t)$ is a lift of γ_t for each $t \in [0,1]$. Hence

$$\operatorname{ind}_{\gamma_t}(0) = \frac{1}{2\pi i}(\widetilde{\gamma}_t(1) - \widetilde{\gamma}_t(0)) = \frac{1}{2\pi i}(\widetilde{H}(1,t) - \widetilde{H}(0,t)).$$

Since this is a continuous function of t with values in $\mathbb{Z}$, it must be constant on $[0,1]$. It follows that $\operatorname{ind}_{\gamma_0}(0) = \operatorname{ind}_{\gamma_1}(0)$ as desired.

$\Leftarrow$: Suppose that $\operatorname{ind}_{\gamma_0}(0) = \operatorname{ind}_{\gamma_1}(0) =: k \in \mathbb{Z}$. Let $\widetilde{\gamma}_0\colon [0,1] \to \mathbb{C}$ be a lift of γ_0 and $\widetilde{\gamma}_1\colon [0,1] \to \mathbb{C}$ be a lift of γ_1. Then

$$\widetilde{\gamma}_0(1) - \widetilde{\gamma}_0(0) = \widetilde{\gamma}_1(1) - \widetilde{\gamma}_1(0) = 2\pi i k.$$

Define $H\colon [0,1]\times[0,1]\to\mathbb{C}^*$ by

$$H(s,t) := \exp((1-t)\widetilde{\gamma}_0(s) + t\widetilde{\gamma}_1(s)), \quad s,t\in[0,1].$$

Then H is a homotopy between γ_0 and γ_1 in $\mathbb{C}^*$; indeed, H is a continuous map with image in $\mathbb{C}^*$, $H(\cdot,0)=\exp(\widetilde{\gamma}_0)=\gamma_0$, and $H(\cdot,1)=\exp(\widetilde{\gamma}_1)=\gamma_1$. Moreover, $H(0,t)=H(1,t)$ for all $t\in[0,1]$, because

$$\begin{aligned}\frac{H(1,t)}{H(0,t)} &= \exp\big((1-t)(\widetilde{\gamma}_0(1)-\widetilde{\gamma}_0(0)) + t(\widetilde{\gamma}_1(1)-\widetilde{\gamma}_1(0))\big)\\ &= \exp((1-t)2\pi ik + t2\pi ik) = \exp(2\pi ik) = 1.\end{aligned}$$

Hence $\gamma_0\sim\gamma_1$ in $\mathbb{C}^*$. □

Corollary 14.13. *Let $\Omega\subseteq\mathbb{C}$ be a region. If $\gamma\colon[0,1]\to\Omega$ a loop that is null-homotopic in Ω, then*

$$\operatorname{ind}_\gamma(z) = 0 \quad \textit{for all } z\in\mathbb{C}\setminus\Omega.$$

Note that the hypothesis of this implication, and hence also the conclusion, is true for all loops γ in a simply connected region Ω.

Proof. Let γ be as in the statement and $z\in\mathbb{C}\setminus\Omega$ be arbitrary. Then γ is homotopic to constant path α in Ω, and hence also in $\mathbb{C}\setminus\{z\}\supseteq\Omega$. Then by Theorem 14.12 we have $\operatorname{ind}_\gamma(z)=\operatorname{ind}_\alpha(z)=0$, where the last equality immediately follows from the definitions. □

Theorem 14.14. *Let $\gamma\colon[0,1]\to\mathbb{C}$ be a loop. Then the function $z\mapsto\operatorname{ind}_\gamma(z)$ is constant on each connected component of $\mathbb{C}\setminus\gamma^*$, and equal to 0 on the unique unbounded component of $\mathbb{C}\setminus\gamma^*$.*

Proof. The set $\mathbb{C}\setminus\gamma^*$ is open, and so a disjoint union of regions, the connected components of $\mathbb{C}\setminus\gamma^*$. Let $D\subseteq\mathbb{C}$ be a sufficiently large open disk with $\gamma^*\subseteq D$. Then $\mathbb{C}\setminus D\subseteq\mathbb{C}\setminus\gamma^*$ is connected and must lie in one of these regions which is necessarily unbounded. All the other regions are contained in D and hence bounded. So there exists precisely one of the connected components of $\mathbb{C}\setminus\gamma^*$ that is unbounded.

Suppose the region Ω is one of the components of $\mathbb{C}\setminus\gamma^*$, and let $z_0,z_1\in\Omega$ be arbitrary. Then there exists a path $\alpha\colon[0,1]\to\Omega$ with $\alpha(0)=z_0$ and $\alpha_1(1)=z_1$ (see Theorem 2.31). Note that then $\alpha^*\cap\gamma^*=\emptyset$. Hence

$$(s,t)\in[0,1]\times[0,1]\mapsto H(s,t) := \gamma(s)-\alpha(t)$$

defines a homotopy between $\gamma - z_0$ and $\gamma - z_1$ in $\mathbb{C}^*$. So by Theorem 14.12 we have

$$\operatorname{ind}_\gamma(z_0) = \operatorname{ind}_{\gamma - z_0}(0) = \operatorname{ind}_{\gamma - z_1}(0) = \operatorname{ind}_\gamma(z_1).$$

This shows that the map $z \mapsto \operatorname{ind}_\gamma(z)$ is constant on each component of $\mathbb{C} \setminus \gamma^*$.

As above, let $D \subseteq \mathbb{C}$ be an open disk with $\gamma^* \subseteq D$. Then D is convex, and so also simply connected. By Corollary 14.13 we have $\operatorname{ind}_\gamma(z) = 0$ for all $z \in \mathbb{C} \setminus D$. By what we have seen, this implies that $\operatorname{ind}_\gamma(z) = 0$ whenever z lies in the unique unbounded component of $\mathbb{C} \setminus \gamma^*$. $\square$

15 Global Cauchy theorems

Proposition 15.1. *Let $[a, b], [c, d] \subseteq \mathbb{R}$ be compact intervals, and $F\colon [a, b] \times [c, d] \to \mathbb{C}$ be a continuous function. Then we have*

$$\int_a^b \left(\int_c^d F(x, y)\, dy \right) dx = \int_c^d \left(\int_a^b F(x, y)\, dx \right) dy. \tag{52}$$

Proof. This is a special case of Fubini's well-known theorem in Measure Theory. One can easily prove this special case directly as follows.

Uniform continuity of F on $R := [a, b] \times [c, d]$ implies that the functions

$$x \mapsto \int_c^d F(x, y)\, dy \quad \text{and} \quad y \mapsto \int_a^b F(x, y)\, dx$$

are continuous on $[a, b]$ and $[c, d]$, respectively. So the integrals in (52) exist.

To establish the equality in (52) one first notes that this is true for a constant function F. The case of an arbitrary continuous function F can easily be derived from this: one subdivides R into small rectangles, and approximates the integrals over F in (52) by sums of integrals of constant functions over the rectangles in the subdivision of R. □

Corollary 15.2. *Let $\alpha, \beta\colon [0, 1] \to \mathbb{C}$ be piecewise smooth paths, and $G\colon \alpha^* \times \beta^* \to \mathbb{C}$ be continuous. Then we have*

$$\int_\alpha \left(\int_\beta G(z, w)\, dw \right) dz = \int_\beta \left(\int_\alpha G(z, w)\, dz \right) dw.$$

Proof. By breaking α and β up into smaller paths if necessary, one may in addition assume that α and β are smooth, and so have continuous derivatives α' and β'. The statement then immediately follows from Proposition 15.1 applied to the function

$$(x, y) \in [0, 1] \times [0, 1] \mapsto F(x, y) := G(\alpha(x), \beta(y))\alpha'(x)\beta'(y).$$

□

Proposition 15.3. *Let $\gamma\colon [0, 1] \to \mathbb{C}$ be a piecewise smooth path, and $f\colon \gamma^* \to \mathbb{C}$ be a continuous function. If we define*

$$F(z) = \int_\gamma \frac{f(w)}{w - z}\, dw \quad \textit{for } z \in U := \mathbb{C} \setminus \gamma^*,$$

then F is holomorphic on U, and $\lim_{|z| \to \infty} |F(z)| = 0$.

Proof. Note that for fixed $z \in \mathbb{C} \setminus \gamma^*$, the function

$$w \mapsto \frac{f(w)}{w - z}$$

is continuous on γ^*; so $F(z)$ is defined.

To establish that F is holomorphic, we want to apply Morera's Theorem (see Theorem 7.13). For this we first show that F is continuous on U.

To see this, let $z_0 \in U$ be arbitrary. Then there exists $r > 0$ such that $|z_0 - w| \geq r$ for all $w \in \gamma^*$. Let $z \in B(z_0, r/2) \subseteq U$ be arbitrary. Then

$$|z - w| \geq |w - z_0| - |z - z_0| \geq r - r/2 = r/2$$

for $w \in \gamma^*$, and so

$$\begin{aligned}
|F(z) - F(z_0)| &\leq \left| \int_\gamma f(w) \left(\frac{1}{w - z} - \frac{1}{w - z_0} \right) dw \right| \\
&\leq \ell(\gamma) \max_{w \in \gamma^*} |f(w)| \frac{|z - z_0|}{|w - z| \cdot |w - z_0|} \\
&\leq |z - z_0| \underbrace{\frac{2}{r^2} \ell(\gamma) \max_{w \in \gamma^*} |f(w)|}_{=:\, M} = M|z - z_0|.
\end{aligned}$$

This inequality implies that F is continuous at z_0.

To verify the other hypothesis in Morera's Theorem, let Δ be an arbitrary oriented triangle with $\Delta \subseteq U = \mathbb{C} \setminus \gamma^*$. If $w \in \gamma^*$, then $w \notin \Delta$, and so w lies in the unbounded component of $\mathbb{C} \setminus \partial\Delta$. So by Theorem 14.14 we have

$$\operatorname{ind}_{\partial\Delta}(w) = 0 \quad \text{for } w \in \gamma^*, \tag{53}$$

where we consider $\partial\Delta$ as a loop (see Subsection 6.2).

Note that the function

$$(z, w) \in \partial\Delta \times \gamma^* \mapsto \frac{f(w)}{w - z}$$

is continuous. Hence by Corollary 15.2 we have

$$\begin{aligned}
\int_{\partial\Delta} F(z)\, dz &= \int_{\partial\Delta} \left(\int_\gamma \frac{f(w)}{w - z}\, dw \right) dz = \int_\gamma \left(\int_{\partial\Delta} \frac{f(w)}{w - z}\, dz \right) dw \\
&= -\int_\gamma f(w) \left(\int_{\partial\Delta} \frac{dz}{z - w} \right) dw \\
&= -2\pi i \int_\gamma f(w) \operatorname{ind}_{\partial\Delta}(w)\, dw = 0 \quad \text{(by Thm. 14.8 and (53))}.
\end{aligned}$$

The holomorphicity of F on U follows.

Finally, we have

$$|F(z)| = \left| \int_\gamma \frac{f(w)}{w-z}\, dw \right| \leq \max_{w \in \gamma^*} |f(w)| \cdot \frac{\ell(\gamma)}{\operatorname{dist}(z, \gamma^*)} \to 0 \quad \text{as } |z| \to \infty.$$

□

15.4. Chains and cycles. Let $\Omega \subseteq \mathbb{C}$ be an open set. A *chain* Γ in Ω is a collection $\gamma_1, \dots, \gamma_n$ of paths in Ω where each of the paths γ_k has an integer $m_k \in \mathbb{Z}$ attached as a weight. We write symbolically

$$\Gamma = m_1\gamma_1 + \cdots + m_n\gamma_n. \tag{54}$$

The chain Γ is called a *cycle* in Ω if each of its path $\gamma_1, \dots, \gamma_n$ is a loop. A chain or a cycle is called *piecewise smooth* if each of its paths has this property. If Γ is as in (54), then we define the *support* of Γ as

$$\Gamma^* := \gamma_1^* \cup \cdots \cup \gamma_n^* \subseteq \mathbb{C}.$$

If Γ is piecewise smooth chain, and $f\colon \Gamma^* \to \mathbb{C}$ is continuous, we define

$$\int_\Gamma f(z)\, dz = m_1 \int_{\gamma_1} f(z)\, dz + \cdots + m_n \int_{\gamma_n} f(z)\, dz.$$

If Γ is a cycle and $z \in \mathbb{C} \setminus \Gamma^*$, we define the *index of* Γ *with respect to the point* z as

$$\operatorname{ind}_\Gamma(z) = m_1 \operatorname{ind}_{\gamma_1}(z) + \cdots + m_n \operatorname{ind}_{\gamma_n}(z).$$

Definition 15.5. Let Ω be an open subset of $\mathbb{C}$. A cycle Γ in Ω is called *null-homologous in* Ω if $\operatorname{ind}_\Gamma(z) = 0$ for all $z \in \mathbb{C} \setminus \Omega$.

Example 15.6. If γ is a null-homotopic loop in an open set $\Omega \subseteq \mathbb{C}$, then the cycle $\Gamma = \gamma$ is null-homologous in Ω. This follows from Corollary 14.13. In particular, every loop in a simply connected region is null-homologous.

More generally, if $\Gamma = m_1\gamma_1 + \cdots + m_n\gamma_n$ is any cycle in a simply connected region $\Omega \subseteq \mathbb{C}$, then Γ is null-homologous in Ω; indeed, by Corollary 14.13 for all $z \in \mathbb{C} \setminus \Omega$ we have

$$\operatorname{ind}_\Gamma(z) = m_1 \underbrace{\operatorname{ind}_{\gamma_1}(z)}_{=0} + \cdots + m_n \underbrace{\operatorname{ind}_{\gamma_n}(z)}_{=0} = 0.$$

Theorem 15.7 (Cauchy's Theorem)**.** *Let $\Omega \subseteq \mathbb{C}$ be open, $f\colon \Omega \to \mathbb{C}$ be a holomorphic function, and Γ be a piecewise smooth cycle that is null-homologous in Ω (i.e., $\operatorname{ind}_\Gamma(z) = 0$ for all $z \in \mathbb{C} \setminus \Omega$).*

Then

$$\operatorname{ind}_\Gamma(z) \cdot f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w - z}\, dw \quad \text{(Cauchy's Integral Formula)} \tag{55}$$

for all $z \in \Omega \setminus \Gamma^*$, and

$$\int_\Gamma f(z)\, dz = 0 \quad \text{(Cauchy's Integral Theorem).} \tag{56}$$

Proof. We will prove (55) first. Define $g\colon \Omega \times \Omega \to \mathbb{C}$ for $(z, w) \in \Omega \times \Omega$ as

$$g(z, w) = \begin{cases} \dfrac{f(w) - f(z)}{w - z} & \text{if } z \neq w, \\[2ex] f'(z) & \text{if } z = w. \end{cases}$$

Then g is continuous on $\Omega \times \Omega$ (exercise!), and so we can define

$$h(z) = \frac{1}{2\pi i} \int_\Gamma g(z, w)\, dw \quad \text{for } z \in \Omega.$$

Then (55) is equivalent to the statement

$$h(z) = 0 \quad \text{for all } z \in \Omega \setminus \Gamma^*. \tag{57}$$

To see this, we first want to show that h is holomorphic on Ω. Again we apply Morera's Theorem. First note that h is continuous on Ω as easily follows from the uniform continuity of g on compact subsets of $\Omega \times \Omega$ (exercise!). If Δ is an arbitrary closed oriented triangle in Ω, then by Corollary 15.2 we have

$$\begin{aligned} \int_{\partial\Delta} h(z)\, dz &= \frac{1}{2\pi i} \int_{\partial\Delta} \left(\int_\Gamma g(z, w)\, dw \right) dz \\ &= \frac{1}{2\pi i} \int_\Gamma \underbrace{\left(\int_{\partial\Delta} g(z, w)\, dz \right)}_{=0} dw = 0. \end{aligned}$$

Here we used Goursat's Lemma (see Theorem 6.3) and the fact that the function $z \mapsto g(z, w)$ is holomorphic on Ω for each fixed $w \in \Omega$ (exercise!). We conclude that $h \in H(\Omega)$.

Let $\widetilde{\Omega} \subseteq \mathbb{C} \setminus \Gamma^*$ be the set of all $z \in \mathbb{C} \setminus \Gamma^*$ for which $\mathrm{ind}_\Gamma(z) = 0$. The set $\widetilde{\Omega}$ is open, because $z \mapsto \mathrm{ind}_\Gamma(z)$ is a locally constant function (as follows from Theorem 14.14). Moreover, $\mathbb{C} \setminus \Omega \subseteq \widetilde{\Omega}$ by our assumption that Γ is null-homologous in Ω. Define

$$\widetilde{h}(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w - z}\, dw \quad \text{for } z \in \widetilde{\Omega}.$$

Then $h \in H(\widetilde{\Omega})$ by Proposition 15.3, and

$$h(z) = \widetilde{h}(z) \quad \text{for all } z \in \Omega \cap \widetilde{\Omega}.$$

So if we define

$$F(z) = \begin{cases} h(z) & \text{for } z \in \Omega, \\ \widetilde{h}(z) & \text{for } z \in \widetilde{\Omega}, \end{cases}$$

then h is well-defined, and F is holomorphic on $\Omega \cup \widetilde{\Omega} = \mathbb{C}$. So F is an entire function. Note that a point $z \in \mathbb{C}$ with $|z|$ sufficiently large lies in the unbounded component of Γ^*. Then $z \in \widetilde{\Omega}$ by Theorem 14.14, and so by Proposition 15.3 we have

$$|F(z)| = |\widetilde{h}(z)| = \left| \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w - z}\, dz \right| \to 0 \quad \text{as } |z| \to \infty.$$

As follows from Liouville's Theorem, we must have $F \equiv 0$; so we have (57) which implies (55).

One easily derives Cauchy's Integral Theorem from this by applying (55) to the function $g \in H(\Omega)$ given by $g(z) = (z - z_0) f(z)$, where $z \in \Omega$ and z_0 is a fixed point in $\Omega \setminus \Gamma^*$; indeed,

$$\int_\Gamma f(w)\, dw = \int_\Gamma \frac{g(w)}{w - z_0}\, dw = \mathrm{ind}_\Gamma(z_0) \cdot g(z_0) = 0.$$

$\square$

Corollary 15.8. *Let $\Omega \subseteq \mathbb{C}$ be a simply connected region, $f \in H(\Omega)$, and γ be a piecewise smooth loop in Ω. Then*

$$\int_\gamma f(z)\, dz = 0. \tag{58}$$

Moreoer, if α and β are two piecewise smooth paths in Ω with the same endpoints, then

$$\int_\alpha f(z)\,dz = \int_\beta f(z)\,dz. \tag{59}$$

Proof. The loop γ is null-homotopic in Ω, and so null-homologous in Ω. Hence (58) follows from Cauchy's Integral Theorem. If $\widetilde{\beta}$ is the path opposite to β, then we get a loop σ by first running through α and then through $\widetilde{\beta}$. So by the first part of the proof we have

$$0 = \int_\sigma f = \int_\alpha f + \int_{\widetilde{\beta}} f = \int_\alpha f - \int_\beta f.$$

□

Corollary 15.9. *Let $\Omega \subseteq \mathbb{C}$ be a simply connected region.*

(a) *Every holomorphic function $f\colon \Omega \to \mathbb{C}$ has a primitive $F \in H(\Omega)$.*

(b) *Every zero-free holomorpic function $f\colon \Omega \to \mathbb{C}$ has a holomorphic logarithm $L \in H(\Omega)$ (i.e., $f = e^L$) (and so a holomorphic square root $S = e^{L/2}$ and other n-th roots).*

Proof. (a) Fix $a \in \Omega$. If $z \in \Omega$ is arbitrary, then there exists a piecewise smooth path α in Ω joining a and z. Define

$$F(z) = \int_\alpha f(w)\,dw.$$

Then F is well-defined by Corollary 15.8. If z' is close to z, then we have

$$F(z') = \int_\alpha f(w)\,dw + \int_{[z,z']} f(w)\,dw,$$

and so

$$\frac{F(z') - F(z)}{z' - z} = \frac{1}{z' - z}\int_{[z,z']} f(w)\,dw.$$

Similarly as in the proof of Corollary 6.4 this implies that $F'(z)$ exists and $F'(z) = f(z)$.

(b) One can construct a logarithm L of f by considering a primitive of $f'/f \in H(\Omega)$. □

Remark 15.10. There are other important facts about simply connected regions in $\mathbb{C}$; for example, we have the Riemann Mapping Theorem: if $\Omega \subseteq \mathbb{C}$ is a simply connected region with $\Omega \neq \mathbb{C}$, then there exists a conformal map from Ω onto $\mathbb{D}$. We will prove this later in this course.

16 Isolated singularities

Definition 16.1. Let $U \subseteq \mathbb{C}$ be open, and $a \in U$. If $f \in H(U \setminus \{a\})$, then f is said to have an *isolated singularity* at a. The singularity is called

(i) *removable* if the function f can be extended to a such that the extended function is holomorphic on U, i.e., if there exists $g \in H(U)$ such that $f = g|U \setminus \{a\}$,

(ii) a *pole* if $\lim_{z \to a} |f(z)| = \infty$,

(iii) an *essential singularity* if it is neither removable nor a pole.

So each isolated singularity is either removable, a pole, or an essential singularity. These cases are mutually exclusive.

Example 16.2. The following functions f are defined and holomorphic on $\mathbb{C} \setminus \{0\}$; so they have 0 as an isolated singularity.

(a) $f(z) = (e^z - 1)/z$. Then 0 is a removable singularity for f; indeed,

$$f(z) = \frac{e^z - 1}{z} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}.$$

The last power series converges for all $z \in \mathbb{C}$, and so it represents an entire function that gives a holomorphic extension of f to the whole complex plane.

(b) $f(z) = 1/z^2$. Then 0 is a pole of f, because $|f(z)| = \dfrac{1}{|z|^2} \to \infty$ as $z \to 0$.

(c) $f(z) = e^{1/z}$. Then 0 is an isolated singularity of f; indeed, if $z_n := 1/n$, then $z_n \to 0$ and $f(z_n) = e^n \to \infty$ as $n \to \infty$; so 0 is not removable. But 0 is not a pole of f either: if $z_n = -1/n$, then $z_n' \to 0$ and $f(z_n') = e^{-n} \to 0$ as $n \to \infty$.

Theorem 16.3. *Let $U \subseteq \mathbb{C}$ be open, $a \in U$, and $f \in H(U \setminus \{a\})$. Then the isolated singularity a of f is removable if and only if f is bounded near a, i.e., there exists $r > 0$ and $M \geq 0$ such that $B(a, r) \subseteq U$ and $|f(z))| \leq M$ for all $z \in B(a, r) \setminus \{a\}$.*

Proof. $\Rightarrow$: This is the easy implication. If a is removable, then f has a holomorphic, and in particular continuous, extension to a; so $\lim_{z\to a} f(z)$ exists which implies that f is bounded near a.

$\Leftarrow$: Suppose that f is bounded near a. Define

$$g(z) = \begin{cases} 0 & \text{for } z = a, \\ (z-a)^2 f(z) & \text{for } z \in U \setminus \{a\}. \end{cases}$$

Then

$$\lim_{z\to a} \frac{g(z) - g(a)}{z - a} = \lim_{z\to a} f(z)(z-a) = 0.$$

So g is differentiable at $a = 0$, and $g'(a) = 0$. Obviously, g is also differentiable at each point $z \in U \setminus \{a\}$. Hence $g \in H(U)$, and so g has a power series expansion

$$g(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

near a. Here $c_n = g^{(n)}(a)/n!$ are the Taylor coefficients of g at a. Since $g(a) = 0 = g'(a)$, we have $c_0 = c_1 = 0$. This implies that

$$f(z) = \sum_{n=2}^{\infty} c_n (z-a)^{n-2} = \sum_{n=0}^{\infty} c_{n+2} (z-a)^n$$

is represented by a power series near a. This shows that f can be holomorphically extended to a by setting $f(a) := c_2$. $\square$

Theorem 16.4. *Let $U \subseteq \mathbb{C}$ be open, $a \in U$, and $f \in H(U \setminus \{a\})$. If a is a pole of f, then there exists a unique number $m \in \mathbb{N}$ (called the* order *or* multiplicity *of the pole), and a unique function $g \in H(U)$ with $g(a) \neq 0$ such that*

$$f(z) = \frac{1}{(z-a)^m} g(z) \quad \text{for } z \in U \setminus \{a\}. \tag{60}$$

If f has a pole of order m at a, then there exist unique numbers $A_1, \ldots, A_m \in \mathbb{C}$, $A_m \neq 0$, and a unique function $h \in H(U)$ such that

$$f(z) = \frac{A_m}{(z-a)^m} + \cdots + \frac{A_1}{(z-a)} + h(z) \quad \text{for } z \in U \setminus \{a\}. \tag{61}$$

The expression

$$P(z) := \frac{A_m}{(z-a)^m} + \cdots + \frac{A_1}{(z-a)}$$

is called the *principal* or *singular part* of the pole a of f.

Proof. Since $|f(z)| \to \infty$ as $z \to a$, the function f has no zeros near a. So the function $z \mapsto 1/f(z)$ is defined in a punctured neighborhood of a and holomorphic there; since

$$\lim_{z\to a} \frac{1}{f(z)} = 0,$$

this function has a removable singularity at a by Theorem 16.3. So there exists a function u that is holomorphic near a such that $u(a) = 0$, and $u = 1/f$ in a punctured neighborhood of a.

Let $m \in \mathbb{N}$ be the order of the zero at a (note that $u \not\equiv 0$); then there exists a function v that is holomorphic near a such that $v(a) \neq 0$ and

$$u(z) = (z-a)^m v(z) \quad \text{for } z \text{ near } a.$$

Then

$$f(z) = \frac{1}{(z-a)^m v(z)} \quad \text{for } z \text{ near } a,\ z \neq a.$$

If we define

$$g(z) = \begin{cases} 1/v(z) & \text{for } z \text{ near } a, \\ f(z)(z-a)^m & \text{for } z \text{ elsewhere in } U, \end{cases}$$

then $g \in H(U)$, $g(a) = 1/v(a) \neq 0$, and (60) is true.

The function g has a power series representation

$$g(z) = \sum_{n=0}^{\infty} a_n(z-a)^n = a_0 + \cdots + a_{m-1}(z-a)^{m-1} + (z-a)^m h(z)$$

near a, where $a_0 = g(a) \neq 0$, and $h \in H(U)$. Dividing by $(z-a)^m$ we obtain (61) with $A_k = a_{m-k}$ for $k = 1, \ldots, m$.

The uniqueness statements are straightforward to prove. For example, suppose that

$$\frac{A_m}{(z-a)^m} + \cdots + \frac{A_1}{(z-a)} + h(z) = f(z) = \frac{B_m}{(z-a)^m} + \cdots + \frac{B_1}{(z-a)} + \widetilde{h}(z)$$

for z near a, $z \neq a$, where $B_1, \ldots, B_m \in \mathbb{C}$, and $\widetilde{h} \in H(U)$. Then one successively proves $A_m = B_m, \ldots, A_1 = B_1$, $h = \widetilde{h}$; indeed, letting $z \to a$ in the identity

$$\begin{aligned} A_m - B_m = (B_{m-1} - A_{m-1})(z-a) + \ldots \\ + (B_1 - A_1)(z-a)^{m-1} + (h(z) - \widetilde{h}(z))(z-a)^m, \end{aligned}$$

we obtain $A_m - B_m = 0$, etc. □

Example 16.5 (Rational functions)**.** Let R be a non-constant rational function written as

$$R(z) = \frac{P(z)}{Q(z)}$$

where P and Q are polynomials, $P, Q \neq 0$. Every zero a of Q is an isolated singularity of R. We can write

$$P(z) = (z-a)^k \widetilde{P}(z) \quad \text{and} \quad Q(z) = (z-a)^l \widetilde{Q}(z),$$

where $k \in \mathbb{N}_0$, $l \in \mathbb{N}$, $\widetilde{P}$ and $\widetilde{Q}$ are polynomials, and $\widetilde{P}(a), \widetilde{Q}(a) \neq 0$. Then

$$\frac{P(z)}{Q(z)} = (z-a)^{k-l} \frac{\widetilde{P}(z)}{\widetilde{Q}(z)},$$

and so a is removable if $k \geq l$, and a pole if $k < l$. By dividing out common factors of P and Q one can assume that R has no removable singularities. Then the only singularities of R are possible poles $a \in \mathbb{C}$ (which are always present unless R is a polynomial). If P and Q have no common factor, then the zeros a of Q are precisely the poles of R.

One can obtain a *partial fraction decomposition* of R as follows. By dividing P by Q with remainder, we can write R in the form

$$R(z) = \frac{P(z)}{Q(z)} = S(z) + \frac{T(z)}{Q(z)},$$

where S and T are polynomials, and $\deg(T) < \deg(Q)$ or $T = 0$. Let us assume that $T \neq 0$, because otherwise R is a polynomial, and that T and Q have no common zeros (otherwise we can cancel common factors). Then the zeros a of Q are precisely the poles of T/Q. Let p_a the denote the principal part of the pole of T/Q at a; then the function

$$z \mapsto \frac{T(z)}{Q(z)} - \sum_{a \text{ zero of } Q} p_a(z)$$

is holomorpic on $\mathbb{C}$ except for finitely many removable singularities corresponding to the zeros of Q; so one can consider it as an entire function. Moreover, it tends to 0 as $|z| \to \infty$. Hence by Liouville's Theorem this function must be identical 0.

The conclusion is that every rational function R can be written as

$$R(z) = S(z) + \sum_{k=1}^{n} \sum_{l=1}^{m_k} \frac{A_{kl}}{(z-a_k)^l} \quad \text{(partial fraction decomposition)},$$

where S is a polynomial, $n \in \mathbb{N}_0$, $a_1, \dots, a_n \in \mathbb{C}$ are the poles of R with their respective orders $m_1, \dots, m_n \in \mathbb{N}$, and the numbers $A_{kl} \in \mathbb{C}$, $l = 1, \dots, m_k$, are coefficients associated with the principal part of each pole a_k, $k = 1, \dots, n$.

Similar considerations show that every holomorphic map $R\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational function (including constant functions, possibly identical equal to $\infty \in \widehat{\mathbb{C}}$). To see this, one may assume that R is non-constant, and that $R(\infty) = 0$ (otherwise one considers R composed with a suitable Möbius transformation). Then the Uniqueness Theorem implies that $R^{-1}(\infty)$ is a finite subset of $\mathbb{C}$. Each point in $R^{-1}(\infty)$ is a pole of R. Then one substracts principal parts, applies Liouville's Theorem, etc.

Theorem 16.6 (Casorati-Weierstrass)**.** *Let $U \subseteq \mathbb{C}$ be open, $a \in U$, and $f \in H(U \setminus \{a\})$. Suppose that a is an essential singularity of f. Then for all $\delta > 0$ such that $B(a, \delta) \subseteq U$, the set $f(B(a, \delta) \setminus \{a\})$ is dense in $\mathbb{C}$, i.e., $\overline{f(B(a, \delta) \setminus \{a\})} = \mathbb{C}$.*

Proof. Suppose not. Then there exists $\delta > 0$ such that $B(a, \delta) \subseteq U$ and $A := \overline{f(B(a, \delta) \setminus \{a\})} \neq \mathbb{C}$. Then the set $\mathbb{C} \setminus A$ is open and non-empty, and so we can pick a point $w \in \mathbb{C} \setminus A$. Moreover, there exists $\epsilon > 0$ such that

$$|f(z) - w| \geq \epsilon \quad \text{for all } z \in B(a, \delta) \setminus \{a\}.$$

Now define

$$g(z) = \frac{1}{f(z) - w} \quad \text{for } z \in B(a, \delta) \setminus \{a\}.$$

Then $g \in H(B(a, \delta) \setminus \{a\})$ and $|g| \leq 1/\epsilon$. By Theorem 16.3 the function g has a removable singularity at a; so the definition of g can be extended to a such that $g \in H(B(a, \delta))$. We can write g in the form

$$g(z) = (z - a)^m \widetilde{g}(z),$$

where $m \in \mathbb{N}_0$, $\widetilde{g} \in H(B(a,\delta))$, and $\widetilde{g}(a) \neq 0$. Then

$$f(z) = w + \frac{1}{g(z)} = w + \frac{1}{(z-a)^m \widetilde{g}(z)}$$

for z near a. This shows that a is a removable singularity (if $m = 0$) or a pole (if $m > 0$) of f, contradicting our hypothesis that a is an essential singularity. $\square$

Remark 16.7. (a) Another version of the Casorati-Weierstrass Theorem is the following statement: if f is a non-constant entire function, then $f(\mathbb{C})$ is dense in $\mathbb{C}$. The proof is very similar. Again one argues by contradiction, and defines an auxilary function g as in the previous proof. Then one invokes Liouville's Theorem to conclude that g is constant, and hence also f. This gives the desired contradiction.

Alternatively, one can analyze the singularity at $a = \infty$ by considering $z \mapsto h(z) = f(1/z)$ which has an isolated singularity at 0; this singularity can only be a pole or an essential singularity. In the first case f must be a polynomial, and we have $f(\mathbb{C}) = \mathbb{C}$ as follows from the Fundamental Theorem of Algebra. In the second case, where 0 is an essential singularity of h, the statement $\overline{f(\mathbb{C})} = \mathbb{C}$ follows from the above version of the Casorati-Weierstrass Theorem.

(b) One can actually prove much stronger statements: if f is as in Theorem 16.6, then there exists $w_0 \in \mathbb{C}$ such that for all small $\delta > 0$ we have $\mathbb{C} \setminus \{w_0\} \subseteq f(B(a,\delta) \setminus \{a\})$. So in every neighborhood of the essential singularity a the function f attains every value with at most one exception. This is known as Picard's "Big" Theorem.

Similarly, if f is a non-constant entire function, then there exists $w_0 \in \mathbb{C}$ such that $\mathbb{C} \setminus \{w_0\} \subseteq \mathbb{C}$, and so f attains every value with at most one exception (Picard's "Little" Theorem).

These theorems will be proved later in this course.

Definition 16.8 (Doubly infinite series)**.** If $c_n \in \mathbb{C}$ are numbers indexed by $n \in \mathbb{Z}$, then we associate with them a *doubly infinite series*, denoted by $\displaystyle\sum_{n=-\infty}^{\infty} c_n$ or $\displaystyle\sum_{n\in\mathbb{Z}} c_n$. The series is said to *converge* if the infinite series $\displaystyle\sum_{n=0}^{\infty} c_n$ and $\displaystyle\sum_{n=-\infty}^{-1} c_n := \sum_{n=1}^{\infty} c_{-n}$ both converge. In this case, $\displaystyle\sum_{n=-\infty}^{\infty} c_n$ also denotes the limit $\sum_{n=0}^{\infty} c_n + \sum_{n=-\infty}^{-1} c_n$, i.e., $\sum_{n=-\infty}^{\infty} c_n = \lim_{N\to\infty} \sum_{n=0}^{N} c_n + \lim_{M\to\infty} \sum_{n=-M}^{-1} c_n$.

The series $\sum_{n=-\infty}^{\infty} c_n$ is said to be *absolutely convergent* if $\sum_{n=-\infty}^{\infty} |c_n|$ converges.

Remark 16.9. If $\sum_{n=-\infty}^{\infty} c_n$ converges, then $c_{\pm n} \to 0$ as $n \to \infty$, and $\sum_{n=-\infty}^{\infty} c_n = \lim_{N\to\infty} \sum_{n=-N}^{N} c_n$. Note that the existence of the last limit does *not* imply the convergence of $\sum_{n=-\infty}^{\infty} c_n$. For example, $\sum_{n=-\infty}^{\infty} n$ does not converge, but the limit $\lim_{N\to\infty} \sum_{n=-N}^{N} n = 0$ exists.

Definition 16.10 (Laurent series)**.** A *Laurent series* is a doubly infinite series of the form $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$, where $a_n \in \mathbb{C}$ for $n \in \mathbb{Z}$ are the *coefficients* of the series and $z_0 \in \mathbb{C}$ its *center*, and $z \in \mathbb{C}$ is a variable point on which the convergence of the Laurent series will depend in general.

We say that the Laurent series *converges uniformly* on a set $M \subseteq \mathbb{C}$ if both series $\sum_{n=-\infty}^{-1} a_n(z-z_0)^n$ and $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converge uniformly for $z \in M$.

Remark 16.11. For every Laurent series $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ there exists an *annulus*

$$A = \{z \in \mathbb{C} : R_1 < |z - z_0| < R_2\},$$

where $R_1, R_2 \in [0, \infty]$, such that the series converges on A and represents a holomorphic series on A, and diverges in the exterior of A ($= \mathbb{C} \setminus \overline{A}$). The set A may be empty, and for $z \in \partial A$ the series may converge or diverge.

To see this, note that

$$\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n = \underbrace{\sum_{n=1}^{\infty} a_{-n}\left(\frac{1}{z-z_0}\right)^n}_{f_1(z)} + \underbrace{\sum_{n=0}^{\infty} a_n(z-z_0)^n}_{f_2(z)}.$$

Let $R_1 := 1/\widetilde{R}_1 \in [0,\infty]$, where $\widetilde{R}_1 \in [0,\infty]$ is the radius of convergence of the power series $\sum_{n=1}^{\infty} a_{-n}w^n$. Then the series $f_1(z)$ converges if

$$|w| = \frac{1}{|z-z_0|} < \widetilde{R}_1 \;\Leftrightarrow\; |z-z_0| > \frac{1}{\widetilde{R}_1} = R_1;$$

similarly, $f_1(z)$ diverges if $|z-z_0| < R_1$.

If $R_2 \in [0,\infty]$ denotes the radius of convergence of $f_2(z)$, then we see that the Laurent series converges on the annulus A as defined above, and diverges in the exterior of A. Moreover, on A both $f_1(z)$ and $f_2(z)$ are holomorphic, which implies that the Laurent series represents a holomorphic function on A.

The convergence properties of power series imply that the Laurent series converges uniformly on compact subsets of A. In particular, if γ is a piecewise smooth path such that the compact set γ^* is contained in A, then the path integral of the Laurent series over the path γ can be obtained by integrating term-by-term.

Theorem 16.12. *Let $R_1, R_2 \in [0,\infty]$, $R_1 < R_2$, and*

$$A := \{z \in \mathbb{C} : R_1 < |z-z_0| < R_2\}.$$

If $f \in H(A)$, then there exists a Laurent series converging on A such that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n \quad \text{for } z \in A. \tag{62}$$

Let $R_1 < r < R_2$, and $\gamma(t) = z_0 + re^{it}$, $t \in [0,2\pi]$. If f has a Laurent series representation as in (62)*, then*

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)^{n+1}}\,dw \quad \text{for } n \in \mathbb{Z}. \tag{63}$$

This formula for the coefficients implies that the Laurent series in (62) is uniquely determined. So every holomorphic function f on an annulus A has a unique Laurent series representation converging on A.

In particular, if z_0 is an isolated singularity of f, then there exists a unique Laurent series representation of f in a punctured neighborhood of z_0.

Proof. Uniqueness: If f has a representation as in (62), then the Laurent series converges uniformly on the compact subset γ^* of A. Hence for all $n \in \mathbb{Z}$, we can evaluate the integral in (63) by integrating term-by-term:

$$\begin{aligned} \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)^{n+1}}\,dw &= \frac{1}{2\pi i}\int_\gamma \left(\sum_{k=-\infty}^{\infty} a_k (w-z_0)^{k-n-1}\right) dw \\ &= \frac{1}{2\pi i}\sum_{k=-\infty}^{\infty} a_k \underbrace{\int_\gamma (w-z_0)^{k-n-1}\,dw}_{=0 \text{ for } k\neq n} \\ &= \frac{1}{2\pi i} a_n \int_\gamma (w-z_0)^{-1}\,dw = \frac{1}{2\pi i} a_n \cdot 2\pi i = a_n. \end{aligned}$$

Existence: If we define a_n as in (63), then the Laurent series in (62) converges for $z \in A$, and represents f on A. To see this, let $z \in A$ be arbitrary. We can choose $r_1, r_2 \in (0,\infty)$ with

$$R_1 < r_1 < |z - z_0| < r_2 < R_2,$$

and define $\gamma_1(t) = z_0 + r_1 e^{it}$ and $\gamma_2(t) = z_0 + r_2 e^{it}$ for $t \in [0, 2\pi]$. We define the cycle Γ as $\Gamma = \gamma_2 - \gamma_1$. Then $\operatorname{ind}_\Gamma(u) = 0$ for $u \in \mathbb{C} \setminus A$, and so Γ is null-homologous in A. Moreover, $\operatorname{ind}_\Gamma(z) = 1$, and so by Cauchy's Integral Formula we have

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w-z}\,dw \\ &= \underbrace{\frac{1}{2\pi i}\int_{\gamma_2} \frac{f(w)}{w-z}\,dw}_{=:f_2(z)} - \underbrace{\frac{1}{2\pi i}\int_{\gamma_1} \frac{f(w)}{w-z}\,dw}_{=:f_1(z)}. \end{aligned} \tag{64}$$

In the same way as in the proof of Theorem 7.1, for $w \in \gamma_2^*$ one writes

$$\frac{f(w)}{w-z} = \frac{f(w)}{(w-z_0)-(z-z_0)} = \frac{f(w)}{(w-w_0)} \cdot \frac{1}{1-\dfrac{(z-z_0)}{(w-w_0)}}$$
$$= \sum_{n=0}^{\infty} (z-z_0)^n \cdot \frac{f(w)}{(w-w_0)^{n+1}},$$

and notes that the convergence of the last series in uniform for $w \in \gamma_2^*$; this follows from

$$\frac{|z-z_0|}{|w-w_0|} = \frac{|z-z_0|}{r_2} < 1 \quad \text{for } w \in \gamma_2^*$$

and the Weierstrass M-test. By integrating term-by-term we conclude that

$$f_2(z) = \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(w)}{w-z}\, dw = \sum_{n=0}^{\infty} b_n (z-z_0)^n, \tag{65}$$

where

$$b_n = \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(w)}{(w-z_0)^{n+1}}\, dw \quad \text{for } n \in \mathbb{N}.$$

To show that $b_n = a_n$, we want to shift the integration to γ in these integrals. Note that the cycle $\gamma_2 - \gamma$ is null-homologous in A, and that the function $w \mapsto \dfrac{f(w)}{(w-z_0)^{n+1}}$ is holomorphic on A for each $n \in \mathbb{N}$. So Cauchy's Integral Theorem implies

$$0 = \int_{\gamma_2 - \gamma} \frac{f(w)}{(w-z_0)^{n+1}}\, dw = 2\pi i (b_n - a_n).$$

So from (65) we conclude that

$$f_2(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n. \tag{66}$$

In a similar way one shows that

$$f_1(z) = -\sum_{n=-\infty}^{-1} a_n (z-z_0)^n. \tag{67}$$

Indeed, for $w \in \gamma_1^*$ we have

$$\begin{aligned}\frac{f(w)}{w-z} &= \frac{f(w)}{(w-z_0)-(z-z_0)} = -\frac{f(w)}{(z-z_0)} \cdot \frac{1}{1-\dfrac{(w-z_0)}{(z-z_0)}} \\ &= -\sum_{n=0}^{\infty}(w-z_0)^n \cdot \frac{f(w)}{(z-w_0)^{n+1}} \\ &= -\sum_{n=-\infty}^{-1}(z-z_0)^n \cdot \frac{f(w)}{(w-w_0)^{n+1}},\end{aligned}$$

and the convergence is uniform on γ_1^*. Integrating term-by-term and shifting the path γ_1 in the integrals to γ based on a similar argument as before, one obtains (67).

By combining (67), (66), and (64), we obtain (62). □

Theorem 16.13. *Let $U \subseteq \mathbb{C}$ be open, $z_0 \in U$, $f \in H(U \setminus \{z_0\})$, and*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$$

be the unique Laurent series expansion of f converging in a punctured neighborhood of z_0. Then z_0 is

(i) *removable if and only if $a_{-n} = 0$ for all $n \in \mathbb{N}$,*

(ii) *a pole of order $m \in \mathbb{N}$ if and only if $a_{-m} \neq 0$ and $a_{-n} = 0$ for all $n \in \mathbb{N}$, $n \geq m+1$,*

(iii) *an essential singularity if and only if $a_{-n} \neq 0$ for infinitely many $n \in \mathbb{N}$.*

In other words, the nature of the isolated singularity can be determined from considering the *singular* or *principal part* $\sum_{n=-\infty}^{-1} a_n(z-z_0)^n$ of the isolated singularity z_0. For a pole this notion of principal part agrees with our previous definition.

Proof. (i) z_0 is removable iff f has a power series expansion near z_0 iff the negative powers of $z - z_0$ in the Laurent series expansion of f are equal to 0 (by the uniqueness of the Laurent series) iff $a_{-n} = 0$ for all $n \in \mathbb{N}$.

(ii) z_0 is a pole of order m iff

$$f(z) = \frac{1}{(z-z_0)^m} \sum_{n=0}^{\infty} b_n (z-z_0)^n \quad \text{near } z_0$$

where the power series is convergent near z_0 and $b_0 \neq 0$ iff

$$f(z) = \sum_{n=-m}^{\infty} a_n (z-z_0)^n \quad \text{near } z_0, \text{ where } a_{-m} \neq 0.$$

(iii) z_0 is an essential singularity iff z_0 is neither removable nor a pole iff $a_{-n} \neq 0$ for infinitely many $n \in \mathbb{N}$. $\square$

Example 16.14. Let $f(z) = \dfrac{1}{(z-2)(z-3)}$. Then $f \in H(\mathbb{C} \setminus \{2,3\})$. What is the Laurent series expansion of f on $A := \{1 < |z-1| < 2\} \subseteq \mathbb{C} \setminus \{2,3\}$?

We first find the partial fraction decomposition of f which must have the form

$$\frac{1}{(z-2)(z-3)} = \frac{A}{z-2} + \frac{B}{z-3}.$$

By multiplying with the denominator of the left-hand side we obtain

$$1 = A(z-3) + B(z-2) = (A+B)z - (3A+2B).$$

By comparing coefficients we get

$$A + B = 0 \quad \text{and} \quad 3A + 2B = -1$$

which implies $A = -1$ and $B = 1$. So for $z \in A$ we have

$$\begin{aligned} f(z) &= \frac{1}{z-3} - \frac{1}{z-2} = \frac{1}{(z-1)-2} - \frac{1}{(z-1)-1} \\ &= -\frac{1}{2} \cdot \frac{1}{1 - \frac{(z-1)}{2}} - \frac{1}{(z-1)} \cdot \frac{1}{1 - \frac{1}{(z-1)}} \\ &= -\frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{2^n} (z-1)^n - \sum_{n=1}^{\infty} \frac{1}{(z-1)^n} = \sum_{n=-\infty}^{\infty} a_n (z-1)^n, \end{aligned}$$

where

$$a_n = \begin{cases} -1/2^{n+1} & \text{for } n \geq 0, \\ -1 & \text{for } n < 0. \end{cases}$$

Remark 16.15. If U is an open subset of a Riemann surface, $a \in U$, and $f\colon U \setminus \{a\} \to \mathbb{C}$ is a holomorphic function, then one calls a an *isolated singularity* of f. One defines a removable singularity, a pole, or an essential singularity in the same way as in Definition 16.1. By using local charts near a one can actually treat this situation in the same way as for functions with isolated singularities in $\mathbb{C}$, and prove analogs of Theorems 16.3, 16.4, and 16.6.

An important special case is when f is a holomorphic function defined in a punctured neighborhood of ∞ on the Riemann sphere. Then ∞ is an isolated singularity of f; its type is the same as the type of the isolated singularity 0 of the map $z \mapsto f(1/z)$. For example, if f is a non-constant polynomial, then ∞ is a pole of f whose order is equal to the degree of the polynomial.

17 The Residue Theorem

Definition 17.1. Let $U \subseteq \mathbb{C}$ be open, $z_0 \in U$, $f \in H(U \setminus \{z_0\})$, and

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

be the unique Laurent series expansion of f converging in a punctured neighborhood of z_0. Then the coefficient a_{-1} is called the *residue of f at z_0* and denoted by $\operatorname{Res}(f, z_0)$.

Proposition 17.2. *Let $U \subseteq \mathbb{C}$ be open, $z_0 \in U$, and $f \in H(U \setminus \{z_0\})$. If f has a pole of order at most $m \in \mathbb{N}$ at z_0, then*

$$\operatorname{Res}(f, z_0) = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \Big((z - z_0)^m f(z) \Big) \Big|_{z=z_0}. \tag{68}$$

Proof. Near z_0 we have $f(z) = \dfrac{1}{(z - z_0)^m} g(z)$, where g is holomorphic; in particular, $z \mapsto (z - z_0)^m f(z) = g(z)$ has a holomorphic extension to $z = z_0$, and it is understood that one uses this holomorphic extension to evaluate the right-hand side of (68).

Now near z_0 the function g has a power series representation

$$g(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n;$$

so $\operatorname{Res}(f, z_0) = b_{m-1} = \dfrac{1}{(m-1)!} g^{(m-1)}(z_0)$. □

Theorem 17.3 (Residue Theorem)**.** *Let $\Omega \subseteq \mathbb{C}$ be an open set, $a_1, \ldots, a_n \in \Omega$ be distinct points, and $f \in H(\Omega \setminus \{a_1, \ldots, a_m\})$. Suppose Γ is a piecewise smooth cycle in Ω that is null-homologous in Ω and for which $a_1, \ldots, a_m \notin \Gamma^*$. Then*

$$\int_\Gamma f(z)\, dz = 2\pi i \sum_{k=1}^{m} \operatorname{ind}_\Gamma(a_k) \cdot \operatorname{Res}(f, a_k). \tag{69}$$

Proof. We can find a small number $r > 0$ such that the disks $\overline{B}(a_k, r)$, $k = 1, \dots, m$, are pairwise disjoint, do not meet Γ^*, and lie in Ω. Define $\gamma_k(t) = a_k + re^{it}$ for $t \in [0, 2\pi]$, and

$$\widetilde{\Gamma} = \Gamma - \operatorname{ind}_\Gamma(a_1)\gamma_1 - \dots - \operatorname{ind}_\Gamma(a_m)\gamma_m.$$

Then $\widetilde{\Gamma}$ is a piecewise smooth cycle in $\widetilde{\Omega} := \Omega \setminus \{a_1, \dots, a_m\}$. For $z \in \mathbb{C} \setminus \Omega$ we have

$$\operatorname{ind}_{\widetilde{\Gamma}}(z) = \operatorname{ind}_\Gamma(z) = 0;$$

moreover, if $k \in \{1, \dots, m\}$, then

$$\operatorname{ind}_{\widetilde{\Gamma}}(a_k) = \operatorname{ind}_\Gamma(a_k) - \operatorname{ind}_\Gamma(a_k) \cdot \underbrace{\operatorname{ind}_{\gamma_k}(a_k)}_{=1} = 0.$$

Hence $\widetilde{\Gamma}$ is null-homologous in $\widetilde{\Omega}$. Since f is holomorphic on $\widetilde{\Omega}$, it follows from Cauchy's Integral Theorem that

$$\int_{\widetilde{\Gamma}} f(z)\,dz = \int_\Gamma f(z)\,dz - \sum_{k=1}^{m} \operatorname{ind}_\Gamma(a_k) \cdot \int_{\gamma_k} f(z)\,dz = 0. \tag{70}$$

Now if $k \in \{1, \dots, m\}$ is fixed and $\displaystyle\sum_{n=-\infty}^{\infty} c_n(z - a_k)^n$ is the Laurent series representation of f near a_k, then this Laurent series converges uniformly for $z \in \gamma_k^*$. Hence

$$\int_{\gamma_k} f(z)\,dz = \sum_{n=-\infty}^{\infty} c_n \int_{\gamma_k} (z - a_k)^n\,dz = 2\pi i c_{-1} = 2\pi i \operatorname{Res}(f, a_k).$$

From this and (70), the relation (69) follows. □

Remark 17.4. In the proof of Theorem 15.7 we derived Cauchy's Integral Theorem from Cauchy's Integral Formula. In the previous argument we used Cauchy's Integral Theorem for the proof of the Residue Theorem. One can complete the circle by noting that the Residue Theorem implies Cauchy's Integral Formula. Actually, if the assumptions are as in Theorem 15.7, then

$$\operatorname{ind}_\Gamma(z)\frac{f^{(n)}(z)}{n!} = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{(w - z)^{n+1}}\,dw \tag{71}$$

for all $z \in \Omega \setminus \Gamma^*$ and $n \in \mathbb{N}_0$. This immediately follows from the Residue Theorem by noting that for $n \in \mathbb{N}_0$ the function

$$g_n(w) := \frac{f(w)}{(w-z)^{n+1}}$$

is holomorphic on $H(U \setminus \{z\})$ and has a pole of order at most $n+1$ at z; so

$$\operatorname{Res}(g_n, z) = \frac{1}{n!}\frac{d^n}{dw^n}\Big((w-z)^{n+1} g_n(w)\Big)\Big|_{w=z} = \frac{f^{(n)}(z)}{n!}.$$

The Residue Theorem is a very powerful tool for the computation of certain integrals.

Example 17.5. $I := \displaystyle\int_0^{2\pi} \frac{dt}{10 + 8\cos t}$. To compute this integral, we let $z = \gamma(t) := e^{it}$ for $t \in [0, 2\pi]$, and convert I into an path integral over the loop γ. So we want to perform the usual substitution backwards. Here (symbolically)

$$dz = ie^{it}\,dt = iz\,dt \quad \text{and so} \quad dt = \frac{dz}{iz}.$$

Moreover, $\cos t = \frac{1}{2}(e^{it} + e^{-it}) = \frac{1}{2}\left(z + \frac{1}{z}\right)$, which gives

$$I = \int_0^{2\pi} \frac{dt}{10 + 8\cos t} = \int_\gamma \frac{1}{10 + 4(z + 1/z)} \cdot \frac{dz}{iz} = \frac{1}{i}\int_\gamma \frac{dz}{4z^2 + 10z + 4}.$$

Now

$$4z^2 + 10z + 4 = 4(z + 1/2)(z + 2).$$

The function $f(z) := \dfrac{1}{(z+1/2)(z+2)}$ is holomorphic on $\mathbb{C}$ except for two poles of order 1 at $-1/2$ and -2. So by the Residue Theorem,

$$\begin{aligned} I &= \frac{1}{4i}\int_\gamma \frac{dz}{(z+1/2)(z+2)} \\ &= \frac{2\pi i}{4i}\Big(\underbrace{\operatorname{ind}_\gamma(-1/2)}_{=1} \cdot \operatorname{Res}(f, -1/2) + \underbrace{\operatorname{ind}_\gamma(-2)}_{=0} \cdot \operatorname{Res}(f, -2)\Big) \\ &= \frac{\pi}{2} \cdot \frac{1}{z+2}\Big|_{z=-1/2} = \frac{\pi}{2} \cdot \frac{1}{3/2} = \frac{\pi}{3}. \end{aligned}$$

Example 17.6.

$$I := \int_0^\infty \frac{dx}{(x^2+1)(x^2+4)} = \lim_{R\to\infty} \int_0^R \frac{dx}{(x^2+1)(x^2+4)}.$$

Then I is a convergent improper integral, and so the last limit exists (which also follows from the argument below).

For fixed $R > 0$ define $\alpha_R(t) = t$ for $t \in [-R, R]$, $\beta_R(t) = Re^{it}$ for $t \in [0, \pi]$, and $\gamma_R = \alpha_R + \beta_R$ (where this chain is considered as a loop obtained from concatenating α_R and β_R). Note that

$$(z^2+1)(z^2+4) = (z+i)(z-i)(z+2i)(z-2i).$$

So the rational function $f(z) = \dfrac{1}{(z^2+1)(z^2+4)}$ has four poles of order 1 at $\pm i$ and $\pm 2i$. For $R > 2$ we have

$$\operatorname{ind}_{\gamma_R}(i) = \operatorname{ind}_{\gamma_R}(2i) = 1 \quad \text{and} \quad \operatorname{ind}_{\gamma_R}(-i) = \operatorname{ind}_{\gamma_R}(-2i) = 0.$$

The Residue Theorem implies that

$$\begin{aligned}\int_{\gamma_R} f(z)\,dz &= 2\pi i \operatorname{Res}(f,i) + 2\pi i \operatorname{Res}(f,2i) \\ &= 2\pi i \cdot \frac{1}{(z+i)(z^2+4)}\bigg|_{z=i} + 2\pi i \cdot \frac{1}{(z^2+1)(z+2i)}\bigg|_{z=2i} \\ &= \frac{2\pi i}{2i\cdot 3} + \frac{2\pi i}{-3\cdot 4i} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}.\end{aligned}$$

Now

$$\int_{\alpha_R} f(z)\,dz = \int_{-R}^R \frac{dx}{(x^2+1)(x^2+4)} = 2\int_0^R \frac{dx}{(x^2+1)(x^2+4)},$$

and

$$\begin{aligned}\left|\int_{\beta_R} f(z)\,dz\right| &\le \ell(\beta_R)\cdot \max_{z\in\beta_R^*} \frac{1}{|z^2+1|\cdot|z^2+4|} \\ &\le \frac{\pi R}{(R^2-1)(R^2-4)} \to 0 \quad \text{as } R\to\infty.\end{aligned}$$

It follows that

$$\begin{aligned} I &= \lim_{R\to\infty}\int_0^R \frac{dx}{(x^2+1)(x^2+4)} = \frac{1}{2}\lim_{R\to\infty}\int_{\alpha_R} f(z)\,dz \\ &= \frac{1}{2}\lim_{R\to\infty}\left(\int_{\gamma_R} f(z)\,dz - \int_{\beta_R} f(z)\,dz\right) \\ &= \frac{\pi}{12} - \frac{1}{2}\lim_{R\to\infty}\int_{\beta_R} f(z)\,dz = \frac{\pi}{12}. \end{aligned}$$

Example 17.7.

$$I := \int_0^\infty \frac{dx}{\sqrt{x}\,(x+1)(x+2)}.$$

This is an improper integral for both integral limits 0 and ∞. It is easy to see that it converges. One can compute I by making the substitution $x = u^2$ and using the method from Example 17.6.

We want to evaluate I based on a different method that applies to a larger class of integrals, for example the integrals obtained by replacing $\sqrt{x}$ in the integrand by a power x^α, $\alpha \in (0,1)$. The main point here is that the square root function cannot be extended to holomorphic function on $\mathbb{C}^*$.

We let $\sqrt{z}$ be the branch of the square root with branch cut along the non-negative real axis, normalized so that $\sqrt{-1} = i$. Then $\sqrt{re^{it}} = \sqrt{r}e^{it/2}$ for $r > 0$, $t \in (0, 2\pi)$.

Fix $R > 2$ and two small numbers $\epsilon > 0$ and $\delta > 0$. We define $\gamma_1(t) = e^{i\delta}t$ for $t \in [\epsilon, R]$, $\gamma_2(t) = Re^{it}$ for $t \in [\delta, 2\pi - \delta]$, $\gamma_3(t) = e^{-i\delta}(R-t)$ for $t \in [0, R-\epsilon]$, and $\gamma_4(t) = \epsilon e^{-it}$ for $t \in [\delta, 2\pi - \delta]$. We consider the piecewise smooth loop $\gamma := \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ obtained from concatenation of these paths. The paths depend on the parameters ϵ, δ, and R, but we suppress this in our notation.

Let

$$f(z) := \frac{1}{\sqrt{z}(z+1)(z+2)} \quad \text{for } z \in \Omega \setminus \{-1, -2\},$$

where $\Omega := \mathbb{C} \setminus [0, \infty)$. Then f is holomorphic on $\Omega \setminus \{-1, -2\}$ and has simple poles at -1 and -2. Moreover, $\operatorname{ind}_\gamma(z) = 0$ for $z \in [0, \infty)$ and so γ is null-homologous in Ω. We have $\operatorname{ind}_\gamma(-1) = \operatorname{ind}_\gamma(-2) = 1$, and

$$\operatorname{Res}(f,-1) = \left.\frac{1}{\sqrt{z}\,(z+2)}\right|_{z=-1} = \frac{1}{\sqrt{-1}\cdot 1} = \frac{1}{i},$$
$$\operatorname{Res}(f,-2) = \left.\frac{1}{\sqrt{z}\,(z+1)}\right|_{z=-2} = \frac{1}{\sqrt{-2}\cdot(-1)} = -\frac{1}{\sqrt{2}\,i}.$$

So by the Residue Theorem,

$$\int_\gamma f(z)\,dz = 2\pi i\big(\operatorname{Res}(f,-1) + \operatorname{Res}(f,-2)\big) = 2\pi\left(1 - \frac{1}{\sqrt{2}}\right).$$

If $x > 0$ and z approaches x from the upper or lower half-plane, then $\sqrt{z} \to \sqrt{x}$ or $\sqrt{z} \to -\sqrt{x}$, respectively. From this one can derive that

$$\lim_{\delta,\epsilon\to 0} \int_{\gamma_1} f(z)\,dz = \int_0^R \frac{dx}{\sqrt{x}\,(x+1)(x+2)},$$

and

$$\lim_{\delta,\epsilon\to 0} \int_{\gamma_3} f(z)\,dz = \int_R^0 \frac{dx}{-\sqrt{x}\,(x+1)(x+2)} = \int_0^R \frac{dx}{\sqrt{x}\,(x+1)(x+2)}.$$

Moreover,

$$\left|\int_{\gamma_4} f(z)\,dz\right| \le \frac{2\pi\epsilon}{\sqrt{\epsilon}\,(1-\epsilon)(2-\epsilon)} \to 0 \quad \text{as } \epsilon \to 0,$$

and

$$\left|\int_{\gamma_2} f(z)\,dz\right| \le \frac{2\pi R}{\sqrt{R}\,(R-1)(R-2)} \to 0 \quad \text{as } R \to 0.$$

It follows that

$$\begin{aligned} 2\pi\left(1 - \frac{1}{\sqrt{2}}\right) = \int_\gamma f &= \lim_{R\to\infty}\left(\lim_{\delta,\epsilon\to 0}\left(\int_{\gamma_1} f + \int_{\gamma_2} f + \int_{\gamma_3} f + \int_{\gamma_4} f\right)\right) \\ &= \lim_{R\to\infty} 2\int_0^R \frac{dx}{\sqrt{x}\,(x+1)(x+2)} \\ &= \int_0^\infty \frac{dx}{\sqrt{x}\,(x+1)(x+2)} = 2I, \end{aligned}$$

and so $I = \pi\big(1 - \frac{1}{\sqrt{2}}\big)$.

Definition 17.8. Let $U \subseteq \mathbb{C}$ be open. A complex-valued function f is called *meromorphic* on U if there exists a set $A \subseteq U$ such that

(i) A consists of isolated points (or equivalently, A has no limit points in the set U),

(ii) $f \in H(U \setminus A)$,

(iii) each point in A is a pole of f.

So a meromorphic function on U is a function that is holomorphic except for isolated singularities that are poles (we may always assume that removable singularities are absent). The possibility $A = \emptyset$ is allowed in the definition; so every holomorphic function on U is also meromorphic.

One can extend a meromorphic function to a holomorphic map $f\colon U \to \widehat{\mathbb{C}}$ by setting $f(a) = \infty$ for a pole a; it is easy to see that in this way the meromorphic functions f on U correspond precisely to the holomorphic maps $f\colon U \to \widehat{\mathbb{C}}$ that are not identically equal to ∞ on any component of U.

Theorem 17.9 (Argument Principle)**.** *Let $U \subseteq \mathbb{C}$ be a region, and f be a non-constant meromorphic function on U. Let $\gamma\colon [0,1] \to U$ be a piecewise smooth loop in U that is null-homologous in U and satisfies $\operatorname{ind}_\gamma(z) \in \{0,1\}$ for each $z \in U \setminus \gamma^*$; in addition assume that there is no zero or pole of f on γ^*.*

Let $\Omega := \{z \in U \setminus \gamma^* : \operatorname{ind}_\gamma(z) = 1\}$, and N_f and P_f be the number of zeros and poles of f in Ω, respectively, accounted according to multiplicity.

Then N_f and P_f are finite, and

$$\operatorname{ind}_\alpha(0) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = N_f - P_f,$$

where $\alpha := f \circ \gamma$.

Proof. First note that

$$\operatorname{ind}_\alpha(0) = \frac{1}{2\pi i}\int_\alpha \frac{dw}{w} = \frac{1}{2\pi i}\int_0^1 \frac{f'(\gamma(t)) \cdot \gamma'(t)}{f(\gamma(t))}\,dt = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz.$$

If a is a zero of f of order $m \in \mathbb{N}$, then there exists a function g holomorphic near a such that $g(a) \neq 0$ and

$$f(z) = (z-a)^m g(z) \quad \text{near } a.$$

Then $f'(z) = m(z-a)^{m-1}g(z) + (z-a)^m g'(z)$, and so

$$\frac{f'(z)}{f(z)} = \frac{m}{z-a} + \frac{g'(z)}{g(z)} \quad \text{near } a.$$

Hence f'/f has a simple pole at a, and

$$\mathrm{Res}(f'/f, a) = m = \text{multiplicity of the zero } a.$$

Similarly, if a is a pole of f, then there exists a function g holomorphic near a such that $g(a) \neq 0$ and

$$f(z) = \frac{1}{(z-a)^m} g(z) \quad \text{near } a.$$

Then

$$f'(z) = -\frac{m}{(z-a)^{m+1}} g(z) + \frac{1}{(z-a)^m} g'(z),$$

and so

$$\frac{f'(z)}{f(z)} = \frac{-m}{z-a} + \frac{g'(z)}{g(z)} \quad \text{near } a.$$

Hence f'/f has a simple pole at a, and

$$\mathrm{Res}(f'/f, a) = -m = -\text{multiplicity of the pole } a.$$

The set $\Omega = \{z \in U \setminus \gamma^* : \mathrm{ind}_\gamma(z) = 1\} \subseteq U$ is open as follows from Theorem 14.14. Let A be the set of zeros and poles of f in Ω, and B be the set of the remaining zeros and poles of f in U. Both A and B consists of isolated points. If $\widetilde{U} := U \setminus B$, then $\widetilde{U}$ is an open set and γ is null-homologous in $\widetilde{U}$.

Note that the set A is finite (which implies that N_f and P_f are finite); indeed, A is a bounded set, because it must be disjoint from the unbounded component of $\mathbb{C} \setminus \gamma^*$; so if A were infinite, then it had a limit point w. Then necessarily $w \in \partial U \subseteq \mathbb{C} \setminus U$, and so $\mathrm{ind}_\gamma(w) = 0$, because γ is null-homologous in U. Then $\mathrm{ind}_\gamma(a) = 0$ for points $a \in A$ close to w by Theorem 14.14. This is impossible as $\mathrm{ind}_\gamma(a) = 1$ for $a \in A$.

Since $f'/f \in H(\widetilde{U} \setminus A)$ we can apply the Residue Theorem and obtain

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \sum_{a \in A} \mathrm{Res}(f'/f, a) = N_f - P_f.$$

$\square$

Theorem 17.10 (Rouché's Theorem)**.** *Let $U \subseteq \mathbb{C}$ be a region, and f and g be non-constant holomorphic functions on U. Let $\gamma\colon [0,1] \to U$ be a piecewise smooth path that is null-homologous in U and satisfies* $\operatorname{ind}_\gamma(z) \in \{0,1\}$ *for each $z \in U \setminus \gamma^*$; in addition assume that there is no zero of f on γ^*.*

Let $\Omega = \{z \in U \setminus \gamma^* : \operatorname{ind}_\gamma(z) = 1\}$, and N_f and N_g be the numbers of zeros of f and g in Ω, respectively, counted according to multiplicity.

If

$$|g(z) - f(z)| < |f(z)| \quad \textit{for all } z \in \gamma^*, \tag{72}$$

then $N_f = N_g$.

So if the functions f and g are sufficiently close on γ^*, then they have the same number of zeros in Ω.

Proof. Note that (72) implies that $g(z) \neq 0$ for $z \in \gamma^*$. Hence we can apply the Argument Principle to both f and g and conclude $N_f = \operatorname{ind}_{f\circ\gamma}(0)$ and $N_g = \operatorname{ind}_{g\circ\gamma}(0)$. The statement will follow from the homotopy invariance of winding numbers if we can show that the loops $f \circ \gamma$ and $g \circ \gamma$ are homotopic in $\mathbb{C}^*$.

We define $H\colon [0,1] \times [0,1] \to \mathbb{C}^*$ by

$$H(s,t) = f(\gamma(s)) + t\big(g(\gamma(s)) - f(\gamma(s))\big) \quad \text{for } s,t \in [0,1].$$

Then H is continuous. By (72) we have

$$|H(s,t)| \geq |f(\gamma(s))| - |g(\gamma(s)) - f(\gamma(s))| > 0$$

for all $s,t \in [0,1]$. So H maps into $\mathbb{C}^*$. Since γ is a loop, and so $\gamma(0) = \gamma(1)$, we have $H(0,t) = H(1,t)$ for all $t \in [0,1]$. So $H(\cdot,t)$ is a loop for all $t \in [0,1]$. Since $H(\cdot,0) = f \circ \gamma$ and $H(\cdot,1) = g \circ \gamma$, the loops $f \circ \gamma$ and $g \circ g$ are indeed homotopic in $\mathbb{C}^*$ as desired. □

Example 17.11. (a) Consider the polynomial $g(z) = z^9 - 2z^6 + z^4 - 9z^2 + 3$. Then one can use Rouché's Theorem to show that g has precisely two zeros (possibly one double zero) in the unit disk $\mathbb{D}$. Indeed, in Theorem 17.10 we let $U := \mathbb{C}$, $\gamma(t) := e^{2\pi i t}$ for $t \in [0,1]$, and $f(z) := -9z^2$. Note that then $\Omega = \{z \in \mathbb{C} \setminus \gamma^* : \operatorname{ind}_\gamma(z) = 1\} = \mathbb{D}$, and that for $z \in \gamma^* = \partial\mathbb{D}$ we have

$$\begin{aligned} |g(z) - f(z)| &= |z^9 - 2z^6 + z^4 + 3| \leq |z|^9 + 2|z|^6 + |z|^4 + 3 = 7 \\ &< 9 = |-9z^2| = |f(z)|. \end{aligned}$$

So N_g, the number of zeros of g in $\mathbb{D}$, is equal to $N_f = 2$, the number of zeros of f in $\mathbb{D}$.

(b) One can derive the Fundamental Theorem of Algebra from Rouché's Theorem. If

$$g(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$$

is a polynomial of degree $n \geq 1$, we let $f(z) = z^n$, and $\gamma(t) = Re^{2\pi it}$ for $t \in [0,1]$. For sufficiently large $R \geq 1$ and $M := |a_{n-1}| + \cdots + |a_0|$ we have for $z \in \gamma^*$,

$$|g(z) - f(z)| = |a_{n-1}z^{n-1} + \cdots + a_0| \leq MR^{n-1} < R^n = |z|^n = |f(z)|.$$

Since f has n zeros in $D(0,R)$, the polynomial g has also $n \geq 1$ zeros.

18 Normal families

Definition 18.1. Let X and Y be metric spaces, and $f\colon X \to Y$ and $f_n\colon X \to Y$ for $n \in \mathbb{N}$ be maps.

(a) We say that f_n *converges to* f *locally uniformly on* X if for all $p \in X$ there exists a neighborhood U of p (i.e., $p \in \operatorname{int}(U)$) such that $f_n|U \to f|U$ uniformly on U.

(b) We say that f_n *converges to* f *compactly on* X or *uniformly on compact subsets of* X if for all compact sets $K \subseteq X$ we have $f_n|K \to f|K$ uniformly on K.

We use the notation $f_n \to f$ locally uniformly on X or $f_n \to f$ compactly on X (as $n \to \infty$).

A metric space X is said to be *locally compact* if each point $p \in X$ has a compact neighborhood $U \subseteq X$.

Proposition 18.2. *Let X and Y be metric spaces, and $f\colon X \to Y$ and $f_n\colon X \to Y$ for $n \in \mathbb{N}$ be maps. Suppose that X is locally compact. Then the following two conditions are equivalent:*

(i) *$f_n \to f$ locally uniformly on X.*

(ii) *$f_n \to f$ compactly on X.*

Moreover, if (i) *or* (ii) *is true and f_n is continuous for each $n \in \mathbb{N}$, then f is also continuous.*

Proof. (i) $\Leftarrow$ (ii): Suppose (ii) is true. Let K be a arbitrary compact subset of X. Then for each $p \in K$ there exists a neighborhood $U_p \subseteq X$ such that $f_n|U_p \to f|U_p$ uniformly on U_p. By compactness of K there exist finitely many points, say $p_1, \dots, p_m \in K$ such that $K \subseteq U_{p_1} \cup \dots \cup U_{p_m}$. Then $f_n \to f$ uniformly on $K \subseteq U_{p_1} \cup \dots \cup U_{p_m}$.

(i) $\Rightarrow$ (ii): Suppose (i) is true, and let $p \in X$ be arbitrary. Since X is locally compact, there exists a compact neighborhood U of p. By our hypothesis, $f_n \to f$ uniformly on U. Hence every point has a neighborhood on which the convergence $f_n \to f$ is uniform. Hence $f_n \to f$ locally uniformly on X.

The last statement follows from a well-known argument. We denote the metric on X and Y by d and ρ, respectively. Suppose that $f_n \to f$ locally uniformly on X, and that f_n is continuous for each $n \in \mathbb{N}$. Let $p \in X$ and $\epsilon > 0$ be arbitrary. We can find a neighborhood $U \subseteq X$ of p such that $f_n \to f$ uniformly on U. Hence there exist $N \in \mathbb{N}$ such that

$$\rho(f(q), f_N(q)) < \epsilon/3 \quad \text{for all } q \in U.$$

Since f_N is continuous at p and U is a neighborhood of p, there exists $\delta > 0$ such that for each point $q \in X$ with $d(p,q) < \delta$ we have $q \in U$ and $\rho(f_N(q), f_N(p)) < \epsilon/3$. For such points q we then have

$$\begin{aligned}\rho(f(q), f(p)) &< \rho(f(q), f_N(q)) + \rho(f_N(q), f_N(p)) + \rho(f_N(p), f(p)) \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.\end{aligned}$$

This shows that f is continuous at p. Since p was arbitrary, f is continuous on X. $\square$

We will be mostly interested in the case where X is an open subset of $\mathbb{C}$ (equipped with the Euclidean metric), or an open subset of $\widehat{\mathbb{C}}$ (equipped with the chordal metric), and $Y = \mathbb{C}$ or $Y = \widehat{\mathbb{C}}$. In these cases, X is locally compact and so the notions of locally uniform and compact convergence are equivalent. We will use this fact repeatedly in the following without mentioning it explicitly.

Example 18.3. (a) Let $f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k$ be a power series with positive radius of convergence $R > 0$. Let $f_n(z) = \sum_{k=0}^{n} a_k(z-z_0)^k$ be the nth partial sum of the power series. We know that $f_n \to f$ uniformly on compact subsets of $B := B(z_0, R)$, and so $f_n \to f$ locally uniformly on B.

(b) Let $f(z) = \sum_{k=-\infty}^{\infty} a_k(z-z_0)^k$ be a Laurent series that converges on an annulus $A = \{z \in \mathbb{C} : R_1 < |z - z_0| < R_2\}$. We write $f(z) = g(z) + h(z)$, where $g(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k$ and $h(z) = \sum_{k=-\infty}^{-1} a_k(z-z_0)^k$.

If $g_n(z) = \sum_{k=0}^{n} a_k(z-z_0)^k$ and $h_n(z) = \sum_{k=-n}^{-1} a_k(z-z_0)^k$ for $n \in \mathbb{N}$, then it follows from (a) and the discussion in Remark 16.11 that $g_n \to g$ and $h_n \to h$ locally uniformly on A. In particular, if $f_n(z) = \sum_{k=-n}^{n} a_k(z-z_0)^k$ for $n \in \mathbb{N}_0$, then $f_n \to f$ locally uniformly on A.

Theorem 18.4. *Let $U \subseteq \mathbb{C}$ be open, $f\colon U \to \mathbb{C}$, and $f_n \in H(U)$ for $n \in \mathbb{N}$. Suppose that $f_n \to f$ locally uniformly on U. Then $f \in H(U)$. Moreover, for each $k \in \mathbb{N}$ we have $f_n^{(k)} \to f^{(k)}$ locally uniformly on U.*

So locally uniform limits of holomorphic functions are holomorphic, and taking derivatives is compatible with taking locally uniform limits.

Proof. The function f is continuous on U as a locally uniform limit of continuous functions (Proposition 18.2). The holomorphicity of f now follows from Morera's Theorem; indeed, if $\Delta \subseteq U$ is an arbitrary closed oriented triangle in U, then $f_n \to f$ uniformly on $\partial\Delta$. Hence

$$\int_{\partial\Delta} f = \lim_{n\to\infty} \underbrace{\int_{\partial\Delta} f_n}_{=0} = 0.$$

For the second statement is enough to establish the case $k = 1$; the general case then follows from this by repeated application.

So let $z_0 \in U$ be arbitrary. Pick $r > 0$ such that $\overline{B}(z_0, r) \subseteq U$. Let $\gamma(t) = z_0 + re^{it}$ for $t \in [0, 2\pi]$. Then for all $z \in \overline{B}(z_0, r/2)$,

$$\begin{aligned}
|f_n'(z) - f'(z)| &= \frac{1}{2\pi}\left|\int_\gamma \frac{f_n(w) - f(w)}{(w-z)^2}\,dw\right| \\
&\leq \frac{r}{(r/2)^2} \max_{w\in\partial B(z_0,r)} |f_n(w) - f(w)| \\
&= \frac{4}{r} \max_{w\in\partial B(z_0,r)} |f_n(w) - f(w)| \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

This estimate implies that $f_n' \to f_n$ uniformly on $\overline{B}(z_0, r/2)$, and it follows that $f_n' \to f$ locally uniformly on U. $\square$

Example 18.5. As in Example 18.3 let $f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k$ be a power series convergent on a disk B, and f_n be the nth partial sum of the power series. Then $f_n \to f$ locally uniformly on B, and so by the last theorem $f'_n \to f'$ locally uniformly on B. This amounts to the statement that a power series can be differentiated term-by-term (already proved in Theorem 5.9); indeed, for $z \in B$, we have

$$\begin{aligned} f'(z) &= \lim_{n\to\infty} f'_n(z) = \lim_{n\to\infty} \frac{d}{dz}\left(\sum_{k=0}^{n} a_k(z-z_0)^k\right) \\ &= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{d}{dz}\left(a_k(z-z_0)^k\right) = \lim_{n\to\infty} \sum_{k=1}^{n} k a_k (z-z_0)^{k-1} \\ &= \sum_{k=1}^{\infty} k a_k (z-z_0)^{k-1} = \sum_{k=0}^{\infty} (k+1) a_{k+1} (z-z_0)^k. \end{aligned}$$

Similarly, a convergent Laurent series can be differentiated term-by-term.

Theorem 18.6 (Hurwitz's Theorem). *Let $U \subseteq \mathbb{C}$ be a region, $f \in H(U)$, and $f_n \in H(U)$ for $n \in \mathbb{N}$. Suppose that $f_n \to f$ locally uniformly on U. If f_n is injective for each $n \in \mathbb{N}$, then f is constant or also injective.*

Proof. We argue by contradiction and assume that under the given hypotheses, f is neither constant nor injective. Then there exist $z_1, z_2 \in U$, $z_1 \neq z_2$, such that $f(z_1) = f(z_2)$. Without loss of generality, we may assume that $f(z_1) = f(z_2) = 0$.

We can choose a small $r > 0$ such that $B_1 := \overline{B}(z_1, r) \subseteq U$, $B_2 := \overline{B}(z_2, r) \subseteq U$, $B_1 \cap B_2 = \emptyset$, and so that f has no zeros on $\partial B_1 \cup \partial B_2$ (a choice satisfying the last condition is possible, because f is non-constant and so the zeros of f are isolated points). Then $m := \inf\{|f(z)| : z \in \partial B_1 \cup \partial B_2\} > 0$.

Since $f_n \to f$ uniformly on $\partial B_1 \cup \partial B_2$, we can find $N \in \mathbb{N}$ such that

$$|f_N(z) - f(z)| < m \leq |f(z)| \quad \text{for all } z \in \partial B_1 \cup \partial B_2.$$

By Rouché's Theorem f_N has the same number of zeros as f in $\mathrm{int}(B_1)$ and in $\mathrm{int}(B_2)$. So f_n has at least one zero, say w_1, in $\mathrm{int}(B_1)$, and one, say w_2 in $\mathrm{int}(B_2)$. Then $w_1 \neq w_2$, and $f_N(w_1) = 0 = f_N(w_2)$. This contradicts the fact that f_N is injective. □

Definition 18.7 (Normal families)**.** Let $U \subseteq \mathbb{C}$ be open, and $\mathcal{F}$ be a family of complex-valued continuous functions on U. We say that $\mathcal{F}$ is a *normal family* if every sequence $\{f_n\}$ of functions in $\mathcal{F}$ has a subsequence $\{f_{n_k}\}$ that converges to some function $f\colon U \to \mathbb{C}$ locally uniformly on U.

We do *not* require that the sublimit f belongs to $\mathcal{F}$.

Definition 18.8. Let $U \subseteq \mathbb{C}$ be open, and $\mathcal{F}$ be a family of complex-valued continuous functions on U. Then the family $\mathcal{F}$ is called

(a) *equicontinuous at a point* $z_0 \in U$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $f \in \mathcal{F}$ and all $z \in U$ we have the implication

$$|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon.$$

(b) *uniformly bounded on* $A \subseteq U$ if there exists $M \geq 0$ such that for all $f \in \mathcal{F}$ and all $z \in A$ we have

$$|f(z)| \leq M.$$

The family is called *uniformly bounded at* $z_0 \in U$ if it is uniformly bounded on $A = \{z_0\}$. It is called *locally uniformly bounded* if every point $z_0 \in U$ has a neighborhood on which $\mathcal{F}$ is uniformly bounded.

The following theorem is a version of the Arzela-Ascoli Theorem. It gives a complete characterization of normal families.

Theorem 18.9 (Arzela-Ascoli)**.** *Let $U \subseteq \mathbb{C}$ be open, and $\mathcal{F}$ be a family of complex-valued continuous functions on U. Then $\mathcal{F}$ is a normal family if and only if $\mathcal{F}$ is equicontinuous and uniformly bounded at each point $z_0 \in U$.*

Proof. $\Rightarrow$: Suppose $\mathcal{F}$ is a normal family. To show that is equicontinuous and uniformly bounded at each point in U we argue by contradiction.

Suppose $\mathcal{F}$ is not uniformly bounded at some point $z_0 \in U$. Then for each $n \in \mathbb{N}$ there exists $f_n \in \mathcal{F}$ such that $|f_n(z_0)| \geq n$. Then the sequence $\{f_n\}$ cannot have a subsequence that converges locally uniformly on U, because no subsequence even has a pointwise limit at z_0. This contradicts the normality of $\mathcal{F}$.

Suppose $\mathcal{F}$ is not uniformly bounded at some point $z_0 \in U$. Then there exists $\epsilon_0 > 0$ ("bad ϵ") such that for each $n \in \mathbb{N}$ there exists $f_n \in \mathcal{F}$ and $z_n \in U$ with $|z_n - z_0| < 1/n$ such that

$$|f_n(z_n) - f_n(z_0)| \geq \epsilon_0 > 0. \tag{73}$$

Since $\mathcal{F}$ is normal we may assume (by passing to a subsequence if necessary) that f_n converges to some continuous function $f\colon U \to \mathbb{C}$ locally uniformly on U. Then $f_n(z_0) \to f(z_0)$ as $n \to \infty$. Since $z_n \to z_0$, the convergence $f_n \to f$ is locally uniform, and f is continuous at z_0, we also have $f_n(z_n) \to f(z_0)$ (excercise!). Hence $|f_n(z_n) - f_n(z_0)| \to 0$ as $n \to \infty$, contradicting (73).

$\Leftarrow$: Suppose $\mathcal{F}$ is equicontinuous and uniformly bounded at each point in U. Let $\{f_n\}$ be an arbitrary sequence in $\mathcal{F}$. In order to establish that $\mathcal{F}$ is a normal family we have to show that $\{f_n\}$ has a subsequence that converges locally uniformly on U.

We choose a countable dense subset $D := \{p_n : n \in \mathbb{N}\}$ in U (for example the set of all points in U whose real and imaginary parts are rational numbers). Since $\mathcal{F}$ is uniformly bounded at each point in U, for each $k \in \mathbb{N}$ the sequence $\{f_n(p_k)\}$ is bounded. Hence we can find a subsequence $\{f_{1n}\}$ of $\{f_n\}$ such that $\{f_{1n}(p_1)\}$ converges. Using the boundedness of $\{f_{1n}(p_2)\}$, we can find a subsequence $\{f_{2n}\}$ of $\{f_{1n}\}$ such that $\{f_{2n}(p_2)\}$ converges. Repeated this argument, for each $k \in \mathbb{N}$ we can find a subsequence $\{f_{kn}\}$ of $\{f_n\}$ such that $\{f_{kn}(p_k)\}$ converges and such that $\{f_{k+1n}\}$ is a subsequence of $\{f_{kn}\}$. Consider the "diagonal" subsequence $\{g_n\}$ given by $g_n := f_{nn}$ for $n \in \mathbb{N}$. Then $\{g_n\}$ is a subsequence of $\{f_n\}$, and, if we disregard finitely many terms, $\{g_n(p_k)\}$ is a subsequence of $\{f_{kn}(p_k)\}$ for each $k \in \mathbb{N}$. Hence $\{g_n(p)\}$ converges for each $p \in D$.

Claim. For each compact set $K \subseteq U$ and each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|g_k(z) - g_n(z)| \leq \epsilon \quad \text{for all } z \in K \text{ and all } k, n \in \mathbb{N} \text{ with } k, n \geq N. \tag{74}$$

To see this, let $\epsilon > 0$ and $K \subseteq U$ be compact. Since $\mathcal{F}$ and hence also the subfamily $\{g_n\}$ is equicontinuous at each point in U, for each $w \in K$ there exists $\delta_w > 0$ such that

$$|g_n(z) - g_n(w)| \leq \epsilon/3 \quad \text{for all } n \in \mathbb{N} \text{ and all } z \in B(w, \delta_w). \tag{75}$$

Since K is compact there exist finitely many points $w_1, \dots, w_m$ such that the balls $B_j := B(w_j, \delta_{w_j})$, $j = 1, \dots, m$, cover K. Since D is dense in U, for each $j \in \{1, \ldots, m\}$ we can pick a point $p'_j \in D \cap B_j$. Then $\{g_n(p'_j)\}$ converges, and so is a Cauchy sequence for each $j = 1, \ldots, m$. Hence there exists $N \in \mathbb{N}$ such that

$$|g_k(p'_j) - g_n(p'_j)| \leq \epsilon/3 \quad \text{whenever } j \in \{1, \ldots, m\} \text{ and } k, n \geq N. \tag{76}$$

Now let $z \in K$ and $k, n \in \mathbb{N}$ with $k, n \geq N$ be arbitrary. Then there exists $j \in \{1, \ldots, m\}$ with $z \in B_j$. Combining (75) and (76) we obtain

$$\begin{aligned} |g_k(z) - g_n(z)| &\leq |g_k(z) - g_k(p'_j)| + |g_k(p'_j) - g_n(p'_j)| + |g_n(p'_j) - g_n(z)| \\ &\leq \epsilon/3 + \epsilon/3 + \epsilon/3. \end{aligned}$$

This establishes the claim.

The claim implies that $\{g_n\}$ converges locally uniformly on U to some limit function. Namely, first it shows that for each $z \in U$ the sequence $\{g_n(z)\}$ is a Cauchy sequence, and hence converges, say $g_n(z) \to g(z) \in \mathbb{C}$ as $n \to \infty$. Moreover, letting $k \to \infty$ in (74) we see that the convergence $g_n \to g$ is uniform on each compact set $K \subseteq U$. So the subsequence $\{g_n\}$ of our original sequence $\{f_n\}$ converges to some limit function locally uniformly on U. Hence $\mathcal{F}$ is a normal family. $\square$

It seems that this theorem completely solves all questions concerning normal families. In practice it often not so easy to establish the relevant equicontinuity and uniform boundedness conditions in order to establish the normality of a family. For example, let $\mathcal{S}$ be the family of all holomorphic functions f on the unit disk $\mathbb{D}$ that are injective and satisfy $f(0) = f'(0) - 1 = 0$. Then one can show that $\mathcal{S}$ is a normal family, but it is not immediately clear how to verify the conditions in the Arzela-Ascoli theorem.

A condition that guarantees that a family is normal is called *a normality criterion*. The following theorem provides the basic normality criterion for holomorphic functions.

Theorem 18.10 (Montel's "Little" Theorem)**.** *Let $U \subseteq \mathbb{C}$ be open, and $\mathcal{F}$ be a family of holomorphic functions on U. Then $\mathcal{F}$ is a normal family if and only if $\mathcal{F}$ is locally uniformly bounded.*

Proof. $\Rightarrow$: This implication follows from the Arzela-Ascoli Theorem (even if we replace the assumption that the functions in the family are holomorphic by the weaker assumption that they are only continuous). Indeed, if the family is normal, then it is equicontinuous and uniformly bounded at each point of U. This in turn implies that the family is locally uniformly bounded on U.

$\Leftarrow$: By the Arzela-Ascoli Theorem it suffices to show that $\mathcal{F}$ is equicontinuous at each point in U. So let $z_0 \in U$ be arbitrary. We can pick $r > 0$ such that $\overline{B}(z_0, r) \subseteq U$ and such that $\mathcal{F}$ is uniformly bounded on $\overline{B}(z_0, r)$. So there exists $M \geq 0$ such that

$$|f(z)| \leq M \quad \text{whenever } f \in \mathcal{F} \text{ and } z \in \overline{B}(z_0, r).$$

Define $\gamma(t) = z_0 + re^{it}$ for $t \in [0, 2\pi]$, and let $z \in B(z_0, r/2)$ and $f \in \mathcal{F}$ be arbitrary. Then

$$\begin{aligned} |f(z) - f(z_0)| &= \frac{1}{2\pi}\left|\int_\gamma f(w)\left(\frac{1}{w-z} - \frac{1}{w-z_0}\right) dw\right| \\ &\leq \frac{1}{2\pi} \cdot 2\pi r \cdot \max_{w \in \partial B(z_0, r)} \frac{|f(w)| \cdot |z - z_0|}{|w-z| \cdot |w - z_0|} \\ &= r \cdot \frac{M}{(r/2)r}|z - z_0| = \frac{2M}{r}|z - z_0|. \end{aligned}$$

Since M is independent of f and z, the equicontiuity of $\mathcal{F}$ at z_0 immediately follows. $\square$

Remark 18.11. One can prove results similar to the one established for holomorphic functions also for meromorphic functions (considered as holomorphic maps into $\widehat{\mathbb{C}}$). In this case one uses the chordal metric in order to measure distances in the target.

One can show that a locally uniform limit of meromorphic functions on a region is again meromorphic (one has to allow the function that is identically equal to ∞ here).

The notion of a *normal family* of continuous maps into $\widehat{\mathbb{C}}$ is defined in the same way as in Definition 18.7. The corresponding version of the Arzela-Ascoli Theorem then says that a family of such maps $\mathcal{F}$ is normal if and only if it is equicontiunous at each point. The proof of this statement is identical to the proof of Theorem 18.9. The uniform boundedness condition is not relevant here since every sequence in $\widehat{\mathbb{C}}$ has a convergent subsequence.

When one considers normal families one has to be careful about the metric used in the target. Every holomorphic function on a region U can also be considered as a map into $\widehat{\mathbb{C}} \supseteq \mathbb{C}$. If a family of holomorphic functions on a region is U is normal as considered as a family of maps into $\mathbb{C}$, then it is also normal as considered as a family of maps into $\widehat{\mathbb{C}}$, but the converse is not true. On can show that a family $\mathcal{F} \subseteq H(U)$ is normal in the latter sense, if and only if every sequence $\{f_n\}$ in $\mathcal{F}$ has a subsequence that converges locally uniformly to the constant function ∞ or to a function holomorphic on U. Often this is taken as the definition of a normal family of holomorphic functions.

One can formulate sufficient conditions for the normality of a family of meromorphic functions that are similar to Montel's Theorem 18.10. A stronger statement is Montel's "Big" Theorem: if each function in a family of meromorphic functions on a region omits three distinct fixed values $a, b, c \in \widehat{\mathbb{C}}$, then it is normal. For example, a family of holomorphic functions omitting 0 and 1 (the third omitted value is ∞ here) is normal (as a family of maps into $\widehat{\mathbb{C}}$).

19 The Riemann Mapping Theorem

Lemma 19.1. *Let $\Omega \subseteq \mathbb{C}$ be a simply connected region, $\Omega \neq \mathbb{C}$. Then there exists a conformal map $g\colon \Omega \to \widetilde{\Omega}$, where $\widetilde{\Omega}$ is a* Koebe region, *i.e., $\widetilde{\Omega}$ is a simply connected region with $0 \in \widetilde{\Omega} \subseteq \mathbb{D}$.*

Proof. Pick $a \in \mathbb{C} \setminus \Omega$. Then $z \mapsto z - a$ is a zero-free holomorphic function on Ω. So by Corollary 15.9 this function has a holomorphic square root S, i.e., $S \in H(\Omega)$ and

$$S(z)^2 = z - a \quad \text{for all } z \in \Omega.$$

The function S is injective on Ω: if $z_1, z_2 \in \Omega$ are arbitrary, then we have the implications

$$\begin{aligned} S(z_1) = S(z_2) &\Rightarrow S(z_1)^2 = S(z_2)^2 \\ &\Rightarrow z_1 - a = z_2 - a \Rightarrow z_1 = z_2. \end{aligned}$$

So S is a holomorphic bijection $S\colon \Omega \to U := S(\Omega)$ onto its image U, and in particular a conformal map (Corollary 9.6). The set U is a region (see Corollary 9.5). It has the following property: if $w \in U$, then $-w \notin U$; indeed, suppose $w, -w \in U$. Then there exist $z_1, z_2 \in \Omega$ such that $w = S(z_1)$ and $-w = S(z_2)$. Hence

$$z_1 - a = S(z_1)^2 = w^2 = S(z_2)^2 = z_2 - a,$$

and so $z_1 = z_2$. This implies

$$w = S(z_1) = S(z_2) = -w,$$

and gives $w = 0$. This is impossible, because the map $z \mapsto z - a = S(z)^2$, and hence also S, is zero-free on Ω.

Since U is open, there exists $w_0 \in U$, and $r > 0$ such that $B(w_0, r) \subseteq U$. Then $B(w_1, r) \cap V = \emptyset$, where $w_1 := -w_0$.

Consider the Möbius transformation $T\colon U \to \mathbb{C}$,

$$T(w) := \frac{r}{w - w_1}.$$

Then $T\colon U \to V := T(U)$ is a conformal map of U onto its image $V = T(U) \subseteq \mathbb{D}$. Pick $u_0 \in V$, and let

$$R(u) := \frac{u - u_0}{1 - \overline{u}_0 u}, \quad u \in \mathbb{D}.$$

Then $R \in \mathrm{Aut}(\mathbb{D})$, and $0 \in \widetilde{\Omega} := R(T(U)) \subseteq \mathbb{D}$. So we have the sequence of holomorphic bijections

$$\Omega \xrightarrow{S} U \xrightarrow{T} V \xrightarrow{R} \widetilde{\Omega} \subseteq \mathbb{D}.$$

Then $g := R \circ T \circ S$ is a holomorphic bijection of Ω onto $\widetilde{\Omega}$, and hence a conformal map from Ω onto $\widetilde{\Omega}$. The set $\widetilde{\Omega}$ is a region with $0 \in \widetilde{\Omega}$. Moreover, $\widetilde{\Omega}$ is simply connected, because this property of a region is invariant under conformal maps. So $\widetilde{\Omega}$ is a Koebe region, and g is a conformal map from Ω onto $\widetilde{\Omega}$. □

If one replaces the condition that the regions in the previous statement are simply connected by the condition that every zero-free holomorphic function on the region has a holomorphic square root, then the previous proof goes through with the conclusion that $\widetilde{\Omega}$ satisfies the latter property. Note this property of a region is invariant under conformal maps of the region.

Lemma 19.2. *Let $\Omega \subseteq \mathbb{D}$ be a Koebe region, and*

$$\mathcal{S}(\Omega) := \{f\colon \Omega \to f(\Omega) \text{ conformal},\ f(\Omega) \subseteq \mathbb{D},\ f(0) = 0\}.$$

Then

$$\alpha := \sup\{|f'(0)| : f \in \mathcal{S}(\Omega)\} \tag{77}$$

is attained as a maximum, i.e., there exists $f_0 \in \mathcal{S}(\Omega)$ such that $|f'(0)| \leq |f_0'(0)| = \alpha$ for all $f \in \mathcal{S}(\Omega)$.

Proof. Note that $\alpha \geq 1$, because $\mathrm{id}_\Omega \in \mathcal{S}(\Omega)$. Let $\{f_n\}$ be a sequence in $\mathcal{S}(\Omega)$ with $|f_n'(0)| \to \alpha$. Montel's Theorem implies that by passing to a subsequence if necessary we may assume that $f_n \to f_0 \in H(\Omega)$ locally uniformly on Ω. Then $f_0(0) = \lim_{n\to\infty} f_n(0) = 0$, and $|f_0'(0)| = \lim_{n\to\infty} |f_n'(0)| = \alpha > 0$. In particular, f_0 is not a constant function. By Hurwitz's Theorem f_0 is injective, and so f_0 is a conformal map of Ω onto $f_0(\Omega) \subseteq \overline{\mathbb{D}}$. Since $f_0(\Omega)$ is open, we must have $f_0(\Omega) \subseteq \mathbb{D}$. Then $f_0 \in \mathcal{S}(\Omega)$ and $|f_0'(0)| = \alpha$ as desired. □

Lemma 19.3. *Let $\Omega \subseteq \mathbb{D}$ be a Koebe region, and $f_0 \in \mathcal{S}(\Omega)$ be a map for which the supremum in (77) is attained. Then $f_0(\Omega) = \mathbb{D}$, i.e., f is a conformal map from Ω onto $\mathbb{D}$.*

Proof. We argue by contradiction; so suppose $U_0 := f_0(\Omega) \neq \mathbb{D}$. Since $0 \in U_0 \subseteq \mathbb{D}$, there exists $a \neq 0$ such that $a \in \mathbb{D} \setminus U_0$. Let R be the Möbius transformation given by

$$R(u) := \frac{u-a}{1-\overline{a}v}.$$

Then $R \in \mathrm{Aut}(\mathbb{D})$, and so R is a conformal map from U_0 onto $U_1 := R(U_0) \subseteq \mathbb{D}$. Then U_1 is a simply connected region. Since $R(a) = 0$, we have $0 \notin U_1$. So by Corollary 15.9 there exists a homolomorphic branch of the square root function on U_1; i.e., a holomorphic map $S\colon U_1 \to \mathbb{C}$ such that

$$S(v)^2 = v \quad \text{for } v \in U_1.$$

Then for $v \in U_1$ we have $|S(v)|^2 = |v| < 1$, and so $U_2 := S(U_1) \subseteq \mathbb{D}$. Moreover, S is injective on U_1; indeed, if for $v_1, v_2 \in \Omega$, we have $S(v_1) = S(v_2)$, then $v_1 = S(v_1)^2 = S(v_2)^2 = v_2$. It follows that S is a conformal map of U_1 onto the simply connected region $U_2 := S(U_1) \subseteq \mathbb{D}$.

Let $b := S(R(0)) \in S(U_1) \subseteq U_2 \subseteq \mathbb{D}$, and T be the Möbius transformation given by

$$T(w) := \frac{w-b}{1-\overline{b}w}.$$

Then $T \in \mathrm{Aut}(\mathbb{D})$, and so T is a conformal map from U_2 onto the simply connected region $U_3 := T(U_2) \subseteq \mathbb{D}$. Note that $0 = T(b) \in T(U_2) = U_3$, and so U_3 is a Koebe region.

We have constructed conformal maps

$$U_0 \xrightarrow{R} U_1 \xrightarrow{S} U_2 \xrightarrow{T} U_3.$$

Consider the composition $g := R \circ S \circ T$. Then g is a conformal map from U_0 onto U_3 with

$$g(0) = (T \circ S \circ R)(0) = T(b) = 0.$$

Moreover,

$$R'(u) = \frac{1-|a|^2}{(1-\overline{a}u)^2}, \quad |S'(v)| = \frac{1}{2\sqrt{|v|}}, \quad T'(w) = \frac{1-|b|^2}{(1-\overline{b}w)^2},$$

and so

$$|R'(0)| = 1-|a|^2, \quad |S'(R(0))| = |S(-a)| = \frac{1}{2\sqrt{|a|}}, \quad |T'(b)| = \frac{1}{1-|b|^2}.$$

Note also that

$$|b|^2 = |S(R(0))^2| = |S(-a)^2| = |-a| = |a| < 1.$$

Putting this all together gives

$$\begin{aligned}|g'(0)| &= \big|T'\big(S(R(0))\big)\big| \cdot |S'(R(0))| \cdot |R'(0)| = |T'(b)| \cdot |S'(-a)| \cdot |R'(0)| \\ &= \frac{1}{1-|b|^2} \cdot \frac{1}{2\sqrt{|a|}} \cdot (1-|a|^2) = \frac{1-|a|^2}{2\sqrt{|a|}\,(1-|a|)} \\ &= \frac{1+|a|}{2\sqrt{|a|}} = 1 + \frac{(1-\sqrt{|a|})^2}{2\sqrt{|a|}} > 1.\end{aligned}$$

Then $f := g \circ f_0$ is a conformal map of Ω onto the Koebe region $g(f_0(\Omega)) = g(U_0) = U_3$, and $f(0) = g(f_0(0)) = g(0) = 0$; so $f \in \mathcal{S}(\Omega)$, but

$$|f'(0)| = |g'(0)| \cdot |f_0'(0)| > |f_0'(0)|.$$

This contradicts the maximality property of f_0. $\square$

Remark 19.4. (a) Again the relevant property of Ω in the previous proof was that every zero-free holomorphic function on Ω has a holomorphic square root (which transfers to $U_0 = f_0(\Omega)$ and all the other regions considered).

(b) The computation in the previous proof make it seem to be a fortunate coincidence that we get the crucial inequality $|g'(0)| > 1$. One can actually see that this inequality must be true almost without any computation if one consider how the hyperbolic length element $ds_h = \dfrac{2|dz|}{1-|z|^2}$ is distorted by the map. If f is a holomorphic map defined near a point $z \in \mathbb{D}$ and has values in $\mathbb{D}$, then this distortion at z is measured by the *derivative of f with respect to the hyperbolic metric* given by

$$D_h(f)(z) := \frac{f^*(ds_h)}{ds_h} = \frac{(1-|z|^2)|f'(z)|}{1-|f(z)|^2}.$$

This derivative has the expected properties (chain rule, behavior for inverse maps, etc.). Note also that if $f(0) = 0$ then $D_h(f)(0) = |f'(0)|$.

We know that $D_h(\varphi)(z) \equiv 1$ if $\varphi \in \mathrm{Aut}(\mathbb{D})$. Moreover, if $f\colon \mathbb{D} \to \mathbb{D}$ is holomorphic, but $f \notin \mathrm{Aut}(\mathbb{D})$, then $D_h(f)(z) < 1$ for all $z \in \mathbb{D}$. So if g is a branch of the inverse f^{-1} defined near a point $z \in \mathbb{D}$, then $D_h(g)(z) =$

$D_h(f)(g(z))^{-1} > 1$. In particular, this is true for the branch S of the square root function (an inverse branch of $z \mapsto z^2$ mapping $\mathbb{D}$ into $\mathbb{D}$) used in the previous proof. This can also verified by direct computation:

$$|S'(z)| = \frac{1}{2\sqrt{|z|}} \quad \text{and} \quad |S(z)|^2 = |z|,$$

which implies

$$D_h(S)(z) = \frac{(1-|z|^2)|S'(z)|}{1-|S(z)|^2} = \frac{1-|z|^2}{2\sqrt{|z|}\,(1-|z|)} = \frac{1+|z|}{2\sqrt{|z|}} > 1,$$

whenever $z \in \mathbb{D}$.

In the previous proof, the function g was given as $g := R \circ S \circ T$, where $R, T \in \mathrm{Aut}(\mathbb{D})$. Moreover, $g(0) = 0$, and so

$$\begin{aligned} |g'(0)| = D_h(g)(0) &= D_h(R)\big(S(T(0))\big) \cdot D_h(S)(T(0)) \cdot D_h(T)(0) \\ &= D_h(S)(T(0)) > 1. \end{aligned}$$

Theorem 19.5 (Riemann Mapping Theorem)**.** *Let $\Omega \subseteq \mathbb{C}$ be a simply connected region, $\Omega \neq \mathbb{C}$. Then there exists a conformal map $f\colon \Omega \to \mathbb{D}$ of Ω onto $\mathbb{D}$.*

If $z_0 \in \Omega$ is arbitrary, then f can be chosen so that

$$f(z_0) = 0 \quad \textit{and} \quad f'(z_0) > 0. \tag{78}$$

Moreover, with this normalization the map f is uniquely determined.

Proof. By Lemma 19.1 there exists a conformal map $h_1\colon \Omega \to \widetilde{\Omega}$ onto a Koebe region $\widetilde{\Omega}$. By Lemma 19.3 there exists a conformal map $h_2\colon \widetilde{\Omega} \to \mathbb{D}$. Then $f := h_2 \circ h_1$ is a conformal map from Ω onto $\mathbb{D}$.

If $z_0 \in \Omega$ is arbitrary, then $w_0 := f(z_0) \in \mathbb{D}$. For fixed $\theta \in [0, 2\pi]$ to be determined momentarily, consider the Möbius transformation

$$\varphi(w) = e^{i\theta}\frac{w - w_0}{1 - \overline{w_0}w}.$$

Then $\varphi \in \mathrm{Aut}(\mathbb{D})$, and $\varphi(w_0) = 0$. So $\widetilde{f} := \varphi \circ f$ is a conformal map from Ω onto $\mathbb{D}$ with $\widetilde{f}(0) = \varphi(w_0) = 0$. Moreover,

$$\widetilde{f}'(z_0) = \varphi'(w_0) \cdot f'(z_0) = e^{i\theta}\frac{f'(z_0)}{1 - |w_0|^2}.$$

By choosing $\theta \in [0, 2\pi]$ appropriately, we get $\widetilde{f}'(z_0) > 0$. Then $\widetilde{f}$ is a conformal map of Ω onto $\mathbb{D}$ with the desired normalization.

Suppose f_1 and f_2 are two conformal maps from Ω onto $\mathbb{D}$ satisfying the normalization (78). Then $\psi := f_2 \circ f_1^{-1} \in \mathrm{Aut}(\mathbb{D})$, $\psi(0) = 0$, and

$$\psi'(0) = f_2'(f_1^{-1}(0)) \cdot (f_1^{-1})'(0) = \frac{f_2'(z_0)}{f_1'(z_0)} > 0.$$

So $\psi(z) = e^{i\alpha} z$, where $\alpha \in [0, 2\pi)$ and $\psi'(0) = e^{i\alpha} > 0$. This implies $\alpha = 0$, $\psi = \mathrm{id}_{\mathbb{D}}$, and $f_1 = f_2$. $\square$

Based on the remarks after Lemmas 19.1 and 19.3 the argument in the previous proof actually shows that if $\Omega \subseteq \mathbb{C}$, $\Omega \neq \mathbb{C}$, is a region such that every zero-free holomorphic function has a holomorphic square root, then Ω is conformally equivalent to $\mathbb{D}$, i.e., there exists a conformal map of Ω onto $\mathbb{D}$.

Example 19.6. In some simple cases the *Riemann map*, i.e., the conformal map of the simply connected region of Ω onto $\mathbb{D}$ can be found explicitly. Here are two examples.

(a) Let $\Omega := \{re^{i\alpha} : 0 < r < 1, 0 < \alpha < \pi/4\}$. The region Ω is convex, and hence simply connected. The Riemann map can be found as a composition of conformal maps $U_0 = \Omega \to U_1 \to U_2 \to U_3 \to U_4 = \mathbb{D}$, where

$$\begin{aligned}
&z \in U_0 \mapsto s := z^4 \in U_1 := \{s \in \mathbb{C} : |s| < 1, \text{ Im } s > 0\},\\
&s \in U_1 \mapsto t := \frac{1+s}{1-s} \in U_2 := \{t \in \mathbb{C} : \text{Re } t > 0, \text{ Im } t > 0\},\\
&t \in U_2 \mapsto u := t^2 \in U_3 := \{u \in \mathbb{C} : \text{Im } u > 0\},\\
&u \in U_3 \mapsto w := \frac{1+iu}{1-iu} \in U_4 := \mathbb{D}.
\end{aligned}$$

So $w = f(z) = \dfrac{1 + i\left(\frac{1+z^4}{1-z^4}\right)^2}{1 - i\left(\frac{1+z^4}{1-z^4}\right)^2}$ gives a conformal map of Ω onto $\mathbb{D}$.

(b) Let $\Omega := \{z \in \mathbb{C} : -\pi < \text{Im } z < \pi\} \setminus (-\infty, 0]$. The region Ω is *starlike with respect to* $p = 1$. This means that $[p, z] \subseteq \Omega$ for all $z \in \Omega$ and implies that Ω is simply connected. Again the Riemann map can be found as a composition of conformal maps $U_0 = \Omega \to U_1 \to U_2 \to U_3 = \mathbb{D}$. Here

$$\begin{aligned}
&z \in U_0 \mapsto u := e^z \in U_1 := \mathbb{C} \setminus (-\infty, 1],\\
&u \in U_1 \mapsto v := \sqrt{u-1} \in U_2 := \{v \in \mathbb{C} : \operatorname{Re} v > 0\} \quad \text{(principal branch)},\\
&v \in U_2 \mapsto w := \frac{1-v}{1+v} \in U_3 := \mathbb{D}.
\end{aligned}$$

So $w = f(z) = \dfrac{1-\sqrt{e^z-1}}{1+\sqrt{e^z-1}}$ gives a conformal map of Ω onto $\mathbb{D}$.

Remark 19.7. Since the upper half-plane $H := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ is conformally equivalent to $\mathbb{D}$, for every simply connected region $\Omega \subseteq \mathbb{C}$, $\Omega \neq \Omega$, there exists a conformal map $f\colon H \to \Omega$. A general class of simply connected regions Ω for which such a map f can be obtained quite explicitly are those bounded by polygons (consisting of straight line segments).

Suppose that $\partial\Omega$ is a finite union of line segments $\partial\Omega = \bigcup_{k=1}^{n} [p_k, p_{k+1}]$ with pairwise disjoint interiors, where $p_{n+1} = p_1$, and let $\alpha_k\pi$ be the interior angle at the corner p_k with $\alpha_k \in (0, 2)$. Then one can show that there exist points $a_1 < a_2 < \cdots < a_n$ on the real axis and constants $A \in \mathbb{C}$ and $B \in \mathbb{C} \setminus \{0\}$ such that f is given by the *Schwarz-Christoffel map*

$$f(z) = A + B \int_{[0,z]} \prod_{k=1}^{n} (\zeta - a_k)^{\alpha_k - 1}\, d\zeta.$$

Here one has to choose suitable branches of the power functions, for example those that attain positive values for large positive values of ζ. Moreover, this conformal map of H onto Ω has a continuous extension to $\overline{H} = \{z \in \mathbb{C} : \operatorname{Im} z \geq 0\} \cup \{\infty\}$ such that $f(a_k) = p_k$ for $k = 1, \ldots, n$.

For example, let $k \in (0, 1)$, and define

$$f(z) = \int_{[0,z]} \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}}$$

for $z \in \overline{H}$. Here the branch of the root is such that for $\zeta = 0$ we have $\sqrt{1} = 1$. One can show that then f is homeomorphism of $\overline{H}$ onto the rectangle

$$R := \{w \in \mathbb{C} : -K \leq \operatorname{Re} w \leq K,\ 0 \leq \operatorname{Im} w \leq K'\}$$

and a conformal map of H onto $\mathrm{int}(R)$. The quantities K and K' are given by *complete elliptic integrals of the first kind:*

$$K = \int_0^{\pi/2} \frac{dt}{\sqrt{1-k^2\sin^2 t}}, \quad \text{and} \quad K' = \int_0^{\pi/2} \frac{dt}{\sqrt{1-k'^2\sin^2 t}},$$

where $k' := \sqrt{1-k^2} \in (0,1)$. The map f sends the points ± 1 and $\pm 1/k$ to the corners of R; more precisely, we have $f(\pm 1) = \pm K$ and $f(\pm 1/k) = \pm K + iK'$.

Theorem 19.8 (Schwarz Reflection Principle)**.** *Let $U^+ \subseteq H^+ := \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$ be open, $U^- := \{\bar{z} : z \in U^+\}$, and $\alpha \subseteq \mathbb{R}$ be open (as a subset of $\mathbb{R}$). Define $\Omega = U^+ \cup \alpha \cup U^-$, and assume that $\alpha \subseteq \mathrm{int}(\Omega)$ (which implies that Ω is open).*

Let $f\colon U^+ \cup \alpha \to \mathbb{C}$ be continuous and holomorphic on U^+. If $f(\alpha) \subseteq \mathbb{R}$, then the extension $F\colon \Omega \to \mathbb{C}$ of f given by

$$F(z) = \begin{cases} f(z) & \text{for } z \in U^+ \cup \alpha, \\ \overline{f(\bar{z})} & \text{for } z \in U^-, \end{cases}$$

is holomorphic on Ω.

If in addition f is injective on $U^+ \cup \alpha$ and $f(U^+) \subseteq H^+$, then F is injective on Ω, and so F is a conformal map of Ω onto its image $F(\Omega)$.

Proof. Obviously, F is continuous on Ω. Moreover, $F|U^-$ is holomorphic on U^-. Indeed, if $a \in U^-$ is arbitrary, then $\bar{a} \in U^+$, and so f has a power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\bar{a})^n \quad \text{for } z \text{ near } \bar{a}.$$

Then

$$F(z) = \overline{f(\bar{z})} = \sum_{n=0}^{\infty} \bar{a}_n (z-a)^n \quad \text{for } z \text{ near } a,$$

and so the holomorphicity of F near a follows.

The holomorphicity of F on the whole set Ω easily follows from its continuity on Ω, its holomorphicity of F on $U^+ \cup U^-$, and Morera's Theorem.

Finally, if f is injective on $U^+ \cup \alpha$ and $f(U^+) \subseteq H^+$, then F is injective on U^- and $F(U^-) \subseteq H^- := \{z \in \mathbb{C} : \mathrm{Im}\, z < 0\}$. This implies that F is injective on Ω. □

Remark 19.9. Using auxiliary Möbius transformations one can easily formulate and prove a more general version of the Schwarz Reflection Principle for meromorphic functions, where the real line in source and target is replaced by two circles C and C'. One then assumes that $\alpha \subseteq C$ and $f(\alpha) \subseteq C'$. If R and S are the reflections in the circles C and C', respectively, then the extension F will be defined as $F(z) = S\big(f(R(z))\big)$ for $z \in U^- := R(U^+)$.

Example 19.10. Let $k \in (0, 1)$, and as in Remark 19.7 define

$$f(z) = \int_{[0,z]} \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta)}}$$

for $z \in \overline{H} = H \cup \widehat{\mathbb{R}} = \{z \in \mathbb{C} : \operatorname{Im} z \geq 0\} \cup \{\infty\}$. Then f maps $\overline{H}$ homeomorphically onto

$$R = \{w \in \mathbb{C} : -K \leq \operatorname{Re} w \leq K,\, 0 \leq \operatorname{Im} w \leq K'\},$$

and H conformally onto $\operatorname{int}(R)$, where K and K' are as in Remark 19.7. Moreover, $f(\pm 1) = \pm K$, and $f(\pm 1/k) = \pm K + iK'$. The inverse map $u \in R \mapsto z \in \overline{H}$ is denoted by $\operatorname{sn}(u; k)$ or simply by $\operatorname{sn} u$ if the parameter k is understood. This function is called the *sinus amplitudinis* (Jacobi's original Latin phrase), the *sine amplitude*, or the *modular sine* (for parameter $k \in (0, 1)$).

Under the this map reflection in the sides of R, and the sides of rectangles obtained by successive reflections, corresponds to reflection in $\widehat{\mathbb{R}}$ on the target side. Using this and applying the Schwarz Reflection Principle repeatedly, one can show that sn can be extended to a meromorphic function on $\mathbb{C}$. From this reflection process it also follows that

$$\operatorname{sn}(u + 4Km + 2iK'n) = \operatorname{sn}(u) \quad \text{for } u \in \mathbb{C},\, m, n \in \mathbb{Z}.$$

So this function has the two periods $4K$ and $2iK$ that are linearly independent over $\mathbb{R}$. Hence it is *doubly periodic*. Functions that are meromorphic on $\mathbb{C}$ and doubly periodic are called *elliptic functions*. So sn is an example of an elliptic function.

20 The Cauchy transform

In this section we use some basic results of measure theory such as Lebesgue's theorem on dominated convergence or Fubini's Theorem. We denote by A Lebesgue measure on $\mathbb{R}^2 \cong \mathbb{C}$, and by dA integration with respect to this measure.

We first fix some notation for function spaces. All functions are complex-valued. Let $U \subseteq \mathbb{C}$ be open. Then $L^1(U)$ and $L^1_{\mathrm{loc}}(U)$ denote the spaces of integrable and locally integrable functions on U, respectively. The space of continuous functions on U is denoted by $C(U)$; we also use this notation if U is an arbitrary metric space. For $k \in \mathbb{N}$ we let $C^k(U)$ be the space of functions on U that have continuous partial derivatives up to order k. As usual $C^\infty(U)$ stands for the space of functions on U that have continuous partial derivatives of all orders. A subscript "c" indicates compact support; so for example, $C_c^\infty(U)$ is the space of all functions $f \in C^\infty(U)$ whose support

$$\operatorname{supp}(f) := \overline{\{z \in U : f(z) \neq 0\}}$$

is a compact subset of U.

Definition 20.1 (Cauchy transform)**.** Let $u \in C_c(\mathbb{C})$. Then the *Cauchy transform* $Tu\colon \mathbb{C} \to \mathbb{C}$ of u is defined as

$$(Tu)(w) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{u(z)}{z-w}\, dA(z) \quad \text{for } w \in \mathbb{C}.$$

Note that the *Cauchy kernel*

$$K(z) := \frac{1}{\pi z}$$

is locally integrable. This implies that the Cauchy transform exists for each $w \in \mathbb{C}$. Actually, $Tu = K * u$, where the *convolution* $K * u$ is given by

$$(K * u)(w) := \int_{\mathbb{C}} K(w-z)u(z)\, dA(z) \quad \text{for } w \in \mathbb{C}.$$

In the following we use z to denote a typical point in $\mathbb{C}$, and use the notation $\dfrac{\partial f}{\partial z}$ and $\dfrac{\partial f}{\partial \bar z}$, or simply f_z and $f_{\bar z}$, for the z- and $\bar z$-derivatives of a function f as defined in Remark 3.13.

The Cauchy transform is in some sense the inverse operator of the differential operator $\dfrac{\partial}{\partial \bar z}$ as the following two lemmas show.

Lemma 20.2. *If $f \in C_c^1(\mathbb{C})$, then $f = T(f_{\bar{z}})$.*

Proof. Note that $f_{\bar{z}} \in C_c(\mathbb{C})$; so $T(f_{\bar{z}})$ is defined.

In order to compute $(Tf_{\bar{z}})(w)$, we may assume that $w = 0$. We introduce polar coordinates in the z-plane: $z = re^{i\alpha}$ and $\bar{z} = re^{-i\alpha}$, where $r \geq 0$ and $\alpha \in [0, 2\pi]$. Then $r = (z \cdot \bar{z})^{1/2}$ and $\alpha = \frac{1}{2i}\log(z/\bar{z})$ (locally for some branch of the logarithm). Then by the chain rule,

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial r} \cdot \frac{\partial r}{\partial \bar{z}} + \frac{\partial}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial \bar{z}}.$$

Here

$$\frac{\partial r}{\partial \bar{z}} = \frac{\partial((z \cdot \bar{z})^{1/2})}{\partial \bar{z}} = \frac{1}{2} \cdot \frac{z^{1/2}}{\bar{z}^{1/2}} = \frac{1}{2} \cdot \frac{(z\bar{z})^{1/2}}{\bar{z}} = \frac{1}{2}e^{i\alpha}$$

(if we choose local branches of the square root appropriately), and

$$\frac{\partial \alpha}{\partial \bar{z}} = \frac{1}{2i}\frac{\partial(\log(z/\bar{z}))}{\partial \bar{z}} = \frac{1}{2i} \cdot \frac{-1}{\bar{z}} = \frac{i}{2r}e^{i\alpha}.$$

So

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}e^{i\alpha} \cdot \frac{\partial}{\partial r} + \frac{i}{2r}e^{i\alpha} \cdot \frac{\partial}{\partial \alpha}.$$

Then by integrating in polar coordinates we obtain

$$\begin{aligned} T(f_{\bar{z}})(0) &= -\frac{1}{\pi}\lim_{\epsilon \to 0}\int_{\mathbb{C}\setminus B(0,\epsilon)} \frac{f_{\bar{z}}(z)}{z}\, dA(z) \quad \text{(by Lebesgue dominated conv.)} \\ &= -\frac{1}{\pi}\lim_{\epsilon \to 0}\int_\epsilon^\infty \int_0^{2\pi} \left(\frac{1}{2}e^{i\alpha}\frac{\partial f(re^{i\alpha})}{\partial r} + \frac{i}{2r}e^{i\alpha}\frac{\partial f(re^{i\alpha})}{\partial \alpha}\right) e^{-i\alpha}\, dr d\alpha \\ &= A + B, \end{aligned}$$

where

$$A = -\frac{1}{2\pi}\lim_{\epsilon \to 0}\int_0^{2\pi} \underbrace{\left(\int_\epsilon^\infty \frac{\partial f(re^{i\alpha})}{\partial r}\, dr\right)}_{= -f(\epsilon e^{i\alpha})} d\alpha = \lim_{\epsilon \to 0}\frac{1}{2\pi}\int_0^{2\pi} f(\epsilon e^{i\alpha})\, d\alpha = f(0)$$

by continuity of f at 0, and

$$B = -\frac{i}{2\pi}\lim_{\epsilon \to 0}\int_\epsilon^\infty \underbrace{\left(\int_0^{2\pi} \frac{1}{r}\frac{\partial f(re^{i\alpha})}{\partial \alpha}\, d\alpha\right)}_{= 0} dr = 0.$$

Here we introduced the limit as $\epsilon \to 0$ in order to avoid possible problems with the factor $1/r$ in the 1-dimensional integrals.

We get $T(f_{\bar{z}})(0) = A + B = f(0)$ as desired. $\square$

Lemma 20.3. *If $u \in C_c^1(\mathbb{C})$, then $Tu \in C^1(\mathbb{C})$, and $(Tu)_{\bar{z}} = u$.*

Proof. The regularity statement is based on some standard fact, and we only give an outline. Namely, if K is a locally integrable kernel on $\mathbb{R}^n$ and u is C^1-smooth on $\mathbb{R}^n$ with compact support, then $K * u$ is C^1-smooth; indeed, if ∂_k denotes the partial derivative with respect to the k-th coordinate on $\mathbb{R}^n$, then we have $\partial_k(K * u) = K * (\partial_k u)$ for $k = 1, \dots, n$. This can easily be proved by considering the relevant difference quotients, and passing to the limit under the integral based on Lebesgue's theorem on dominated convergence.

If K denotes the Cauchy kernel, then in our situation we have $Tu = K * u \in C^1(\mathbb{C})$. Moreover, since the operator $\partial_{\bar{z}}$ is a linear combination of the usual partial differential operators on $\mathbb{R}^2 \cong \mathbb{C}$, we conclude

$$(Tu)_{\bar{z}} = (K * u)_{\bar{z}} = K * u_{\bar{z}} = T(u_{\bar{z}}) = u,$$

where we used Lemma 20.2 in the last step. $\square$

Lemma 20.4. (a) *Let $\gamma\colon [0,1] \to \mathbb{C}$ be a C^1-smooth path, and $u \in C_c(\mathbb{C})$. Define $F\colon \mathbb{C} \times [0,1] \to \mathbb{C}$ by*

$$F(z,t) = \begin{cases} \dfrac{u(z) \cdot \gamma'(t)}{z - \gamma(t)} & \text{for } z \neq \gamma(t), \\ 0 & \text{for } z = \gamma(t). \end{cases}$$

Then F is integrable on $\mathbb{C} \times [0,1]$ with respect to the product of Lebesgue measures on $\mathbb{C}$ and on $[0,1]$.

(b) *Let Γ be a piecewise smooth cycle in $\mathbb{C}$. Then the function $z \in \mathbb{C} \mapsto \operatorname{ind}_\Gamma(z)$ is almost everywhere defined and integrable.*

Proof. (a) The exceptional set $E := \{(z,t) \in \mathbb{C} \times [0,1] : z = \gamma(t)\}$ is closed, and hence a Borel set. Moreover, F is continuous on the complement of E. From this it follows that F is a Borel function, and hence measurable.

We can pick constants $R > 0$ and $M \geq 0$ such that the sets $\operatorname{supp}(u)$ and γ^* both lie in $\overline{B}(0,R)$, and $|u(z)| \leq M$ for $z \in \overline{B}(0,R)$.

Then we have

$$\begin{aligned}\int_0^1 \left(\int_{\mathbb{C}} |F(z,t)|\, dA(z) \right) dt &\le \int_0^1 \left(\int_{\mathbb{C}} \frac{|u(z)| \cdot |\gamma'(t)|}{|z-\gamma(t)|}\, dA(z) \right) dt \\ &\le M \int_0^1 \left(\int_{\overline{B}(0,R)} \frac{|\gamma'(t)|}{|z-\gamma(t)|}\, dA(z) \right) dt \\ &\le M \int_0^1 \left(\int_{\overline{B}(0,2R)} \frac{|\gamma'(t)|}{|u|}\, dA(u) \right) dt \\ &= M \left(\int_{\overline{B}(0,2R)} \frac{1}{|u|}\, dA(u) \right) \left(\int_0^1 |\gamma'(t)|\, dt \right) < \infty.\end{aligned}$$

This implies that F is integrable on $\mathbb{C} \times [0,1]$.

(b) If γ is a smooth path, then a simple covering argument shows that $A(\gamma^*) = 0$. This implies that $A(\Gamma^*) = 0$. The function $z \mapsto \mathrm{ind}_\Gamma(z)$ is defined on $\mathbb{C} \setminus \Gamma^*$, and hence almost everywhere. Since this function is locally constant on $\mathbb{C} \setminus \Gamma^*$ as follows from Theorem 14.14, it is measurable.

To show that it is also integrable, we may assume that Γ consists of one loop α, and represent the winding number as an integral as in Theorem 14.8. Pick $R > 0$ large enough so that $\alpha^* \subseteq B(0,R)$. Then $\mathbb{C} \setminus B(0,R)$ lies in the unbounded component of $\mathbb{C} \setminus \alpha^*$, and so $\mathrm{ind}_\alpha(z) = 0$ for $z \in \mathbb{C} \setminus B(0,R)$ (Theorem 14.14). By breaking up the loop α into finitely many C^1-smooth paths, we are reduced to proving that if $\gamma\colon [0,1] \to B(0,R)$ is a C^1-smooth path and if we define

$$h(z) = \int_\gamma \frac{d\zeta}{\zeta - z} = \int_0^1 \frac{\gamma'(t)}{\gamma(t) - z}\, dt$$

for $z \in B(0,R)$, then $h \in L^1(B(0,R))$. Note that this function is defined on $B(0,R) \setminus \gamma^*$, and hence almost everywhere on $B(0,R)$. It is easy to see that it is continuous on $B(0,R) \setminus \gamma^*$, and so measurable.

The integrability of h follows from part (a); indeed, we can pick a function $u \in C_c(\mathbb{C})$ such that $u(z) = 1$ for $z \in B(0,R)$. If F is defined as in (a), then by Fubini's Theorem the function $z \mapsto \int_0^1 |F(z,t)|\, dt$ is almost everywhere defined and integrable on $\mathbb{C}$. Hence

$$\begin{aligned}\int_{B(0,R)} |h(z)|\, dA(z) &= \int_{B(0,R)\setminus\gamma^*} |h(z)|\, dA(z) \\ &= \int_{B(0,R)\setminus\gamma^*} \left| \int_0^1 \frac{\gamma'(t)}{\gamma(t)-z}\, dt \right| dA(z) \\ &= \int_{B(0,R)\setminus\gamma^*} \left(\int_0^1 \frac{|u(z)|\cdot|\gamma'(t)|}{|\gamma(t)-z|}\, dt \right) dA(z) \\ &\le \int_{\mathbb{C}} \left(\int_0^1 |F(z,t)|\, dt \right) dA(z) < \infty.\end{aligned}$$

□

Proposition 20.5 (Gauss-Green Formula; preliminary version)**.** *Let $u \in C^1_c(\mathbb{C})$, and Γ be a piecewise smooth cycle in $\mathbb{C}$. Then*

$$\frac{1}{2i}\int_\Gamma u(z)\, dz = \int_{\mathbb{C}} \operatorname{ind}_\Gamma(z)\cdot u_{\bar z}(z)\, dA(z).$$

Note that $u_{\bar z}$ is a continuous function with compact support. This and Lemma 20.4 (b) imply that the integrand on the right hand side is an almost everywhere defined and integrable function.

Proof. By Lemma 20.2 we know that

$$u(w) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{u_{\bar z}(z)}{z-w}\, dA(z)$$

for all $w \in \mathbb{C}$. Integrating over Γ and interchanging the order of integration (which is justified by Fubini's Theorem and Lemma 20.4 (a) applied to paths obtained from splitting up the loops in Γ into C^1-smooth pieces), we obtain

$$\begin{aligned}\frac{1}{2i}\int_\Gamma u(w)\, dw &= -\frac{1}{2\pi i}\int_\Gamma \left(\int_{\mathbb{C}} \frac{u_{\bar z}(z)}{z-w}\, dA(z) \right) dw \\ &= -\frac{1}{2\pi i}\int_{\mathbb{C}} \left(\int_\Gamma \frac{u_{\bar z}(z)}{z-w}\, dw \right) dA(z) \\ &= \int_{\mathbb{C}} \operatorname{ind}_\Gamma(z)\cdot u_{\bar z}(z)\, dA(z).\end{aligned}$$

□

20.6. Partitions of unity. Let $V \subseteq \mathbb{C}$ be open, and $\mathcal{U} = \{U_i \in I\}$ be a cover of V by open sets $U_i \subseteq V$ indexed by a set I. We need the following two statements about partitions of unity.

Version I ("with compact supports"). There exists a countable family $\{\varphi_k : k \in \mathbb{N}\}$ of functions such that

(i) $\varphi_k \in C^\infty(V)$, $0 \le \varphi_k \le 1$, and $\operatorname{supp}(\varphi_k) \subseteq V$ is compact for all $k \in \mathbb{N}$,

(ii) the supports $\operatorname{supp}(\varphi_k)$, $k \in \mathbb{N}$, form a *locally finite* family, i.e., every point $p \in V$ has a neighborhood N such that $\operatorname{supp}(\varphi_k) \cap N \neq \emptyset$ for only finitely many $k \in \mathbb{N}$,

(iii) the supports $\operatorname{supp}(\varphi_k)$, $k \in \mathbb{N}$, form a family that is *subordinate* to $\mathcal{U}$, i.e., for every $k \in \mathbb{N}$ there exists $i(k) \in I$ with $\operatorname{supp}(\varphi_k) \subseteq U_{i(k)}$,

(iv) $\displaystyle\sum_{k \in \mathbb{N}} \varphi_k(z) = 1$ for all $z \in V$.

Note that in the last sum only finitely many terms are non-zero near each point by property (ii). So locally we can treat this as a finite sum and take partial derivatives term-by-term, for example.

The phrase "partition of unity" is of course explained by the identity (iv). It often allows one to break up a given function into "bumps" with well-localized support.

Version II ("with same index set"). There exists a family $\{\varphi_i : i \in I\}$ of functions having the above properties (i)–(iv) with the following modifications: the functions φ_i do not necessarily have compact support, and (iii) is replaced by $\operatorname{supp}(\varphi_i) \cap V \subseteq U_i$ for all $i \in I$.

So in Version I the functions of the partition of unity have compact support, while in Version II we can index the family by the same index set as the cover $\mathcal{U}$. This is not always possible if one insists on compact support of the functions.

Lemma 20.7. *Let $U \subseteq \mathbb{C}$ be open, and $K \subseteq U$ be compact. Then there exists $\varphi \in C_c^\infty(U)$ such that $0 \le \varphi \le 1$, $\varphi(z) = 1$, and $\nabla\varphi(z) = 0$ for all $z \in K$.*

Proof. We can choose an open set $V \subseteq \mathbb{C}$ such that $\overline{V}$ is compact and $K \subseteq V \subseteq \overline{V} \subseteq U$. Then $\mathcal{U} = \{U \setminus K, V\}$ is an open cover of U. Hence there exists a partition of unity on V subordinate to $\mathcal{U}$; more precisely, there exist two functions $\varphi, \psi \in C^\infty(U)$ with $0 \le \varphi, \psi \le 1$, $\operatorname{supp}(\varphi) \cap U \subseteq V$, $\operatorname{supp}(\psi) \cap U \subseteq U \setminus K$, and $\varphi + \psi = 1$ on U. Then φ vanishes on $U \setminus V$, and so $\operatorname{supp}(\varphi) \subseteq \overline{V}$. This implies $\varphi \in C_c^\infty(U)$.

Moreover, $\operatorname{supp}(\psi) \cap K = \emptyset$, and so $\psi(z) = 0$ for all points z in an open neighborhood of K. So for $z \in K$ we have $\varphi(z) = 1 - \psi(z) = 1$, and $\nabla\varphi(z) = -\nabla\psi(z) = 0$. $\square$

Theorem 20.8. *Let $\Omega \subseteq \mathbb{C}$ be open, $f \in C^1(\Omega)$, and Γ be a piecewise smooth cycle in Ω that is null-homologous in Ω. Then*

$$\frac{1}{2i}\int_\Gamma f(z)\,dz = \int_\Omega \operatorname{ind}_\Gamma(z)\cdot f_{\bar z}(z)\,dA(z) \quad \text{(Gauss-Green Formula)},$$

and for all $z_0 \in \Omega \setminus \Gamma^*$,

$$\operatorname{ind}_\Gamma(z_0)\cdot f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z - z_0}\,dz - \frac{1}{\pi}\int_\Omega \operatorname{ind}_\Gamma(z)\cdot\frac{f_{\bar z}(z)}{z - z_0}\,dA(z)$$

(Cauchy-Green Formula or Pompeiu's Formula).

Proof. We have $\operatorname{ind}_\Gamma(z) = 0$ for all $z \in \mathbb{C}$ near $\partial\Omega \subseteq \mathbb{C} \setminus \Omega$ (since Γ is null-homologous in Ω), and also $\operatorname{ind}_\Gamma(z) = 0$ for all $z \in \mathbb{C}$ with $|z|$ large. Hence we can pick a compact set $K \subseteq \Omega$ such that $\Gamma^* \subseteq K$ and $\operatorname{ind}_\Gamma(z) = 0$ for all $z \in \mathbb{C} \setminus K$. Actually, one can take K of the form

$$K = \{z \in \Omega : \operatorname{dist}(z, \partial\Omega) \ge \delta \text{ and } |z| \le R\},$$

where $\delta > 0$ is small enough and $R > 0$ is large enough.

By Lemma 20.7 there exists $\varphi \in C_c^\infty(\Omega)$ with $0 \le \varphi \le 1$ such that $\varphi(z) - 1 = \varphi_{\bar z}(z) = 0$ for $z \in K$. We now apply Proposition 20.5 to $u := \varphi f$ (considered as a function on $\mathbb{C}$ by setting it $\equiv 0$ on $\mathbb{C} \setminus \Omega$). Then $u \in C_c^1(\mathbb{C})$, $u(z) = f(z)$ for $z \in \Gamma^* \subseteq K$, and

$$u_{\bar z} = \varphi_{\bar z} f + \varphi f_{\bar z} = f_{\bar z} \quad \text{on } K.$$

Hence

$$\begin{aligned}\frac{1}{2i}\int_\Gamma f(z)\,dz &= \frac{1}{2i}\int_\Gamma u(z)\,dz = \int_{\mathbb{C}} \operatorname{ind}_\Gamma(z)\cdot u_{\bar z}(z)\,dA(z)\\ &= \int_K \operatorname{ind}_\Gamma(z)\cdot u_{\bar z}(z)\,dA(z) = \int_K \operatorname{ind}_\Gamma(z)\cdot f_{\bar z}(z)\,dA(z)\\ &= \int_\Omega \operatorname{ind}_\Gamma(z)\cdot f_{\bar z}(z)\,dA(z).\end{aligned}$$

For the proof of the Cauchy-Green Formula we let $\widetilde{\Gamma} := \Gamma - \mathrm{ind}_\Gamma(z_0)\gamma$, where $\gamma(t) := z_0 + re^{it}$ for $t \in [0, 2\pi]$ and $r > 0$ is small. Here γ depends on r, but we suppress this in our notation.

Define

$$g(z) = \frac{1}{\pi}\frac{f(z)}{z - z_0}$$

for $z \in \widetilde{\Omega} := \Omega \setminus \{z_0\}$. Then $g \in C^1(\widetilde{\Omega})$ and $\widetilde{\Gamma}$ is null-homologous in $\widetilde{\Omega}$. Note that

$$g_{\bar z}(z) = \frac{1}{\pi} \cdot \frac{\partial}{\partial \bar z}\left(\frac{f(z)}{z - z_0}\right) = \frac{1}{\pi} \cdot \frac{f_{\bar z}(z)}{z - z_0} \quad \text{for } z \in \widetilde{\Omega}.$$

By the first part we have,

$$\begin{aligned}\frac{1}{2i}\int_{\widetilde{\Gamma}} g(z)\,dz &= \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z - z_0}\,dz - \mathrm{ind}_\Gamma(z_0) \cdot \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}\,dz \qquad (79)\\ &= \int_{\widetilde{\Omega}} \mathrm{ind}_{\widetilde{\Gamma}}(z) \cdot g_{\bar z}(z)\,dA(z)\\ &= \frac{1}{\pi}\int_\Omega \mathrm{ind}_\Gamma(z) \cdot \frac{f_{\bar z}(z)}{z - z_0}\,dA(z) - \mathrm{ind}_\Gamma(z_0) \cdot \frac{1}{\pi}\int_\Omega \mathrm{ind}_\gamma(z) \cdot \frac{f_{\bar z}(z)}{z - z_0}\,dA(z)\\ &= \frac{1}{\pi}\int_\Omega \mathrm{ind}_\Gamma(z) \cdot \frac{f_{\bar z}(z)}{z - z_0}\,dA(z) - \mathrm{ind}_\Gamma(z_0) \cdot \frac{1}{\pi}\int_{B(z_0,r)} \frac{f_{\bar z}(z)}{z - z_0}\,dA(z),\end{aligned}$$

where in the last step we used that

$$\mathrm{ind}_\gamma(z) = \begin{cases} 0 & \text{for } z \in \mathbb{C} \setminus \overline{B}(z_0, r),\\ 1 & \text{for } z \in B(z_0, r).\end{cases}$$

Note that

$$\lim_{r\to 0} \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}\,dz = \lim_{r \to 0} \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\,dt = f(z_0)$$

by continuity of f at $z_0 \in \Omega$, and

$$\lim_{r\to 0}\int_{B(z_0,r)} \frac{f_{\bar z}(z)}{z - z_0}\,dA(z) = 0$$

as easily follows from Lebesgue's theorem on dominated convergence.

If we let $r \to 0$ in (79) and use these limit identities, then the Cauchy-Green formula follows. $\square$

Corollary 20.9 (Area formula). *Let γ be a piecewise smooth loop in $\mathbb{C}$ and assume that $\operatorname{ind}_\gamma(z) \in \{0, 1\}$ for all $z \in \mathbb{C} \setminus \gamma^*$. Define*

$$\Omega = \{z \in \mathbb{C} \setminus \gamma^* : \operatorname{ind}_\gamma(z) = 1\}.$$

Then

$$A(\Omega) = \frac{1}{2i} \int_\gamma \bar{z}\, dz.$$

Proof. Note that Ω is an open set, and hence measurable. We apply the Gauss-Green Formula on $\mathbb{C}$, where $\Gamma = \gamma$ and $f(z) = \bar{z}$ for $z \in \mathbb{C}$. Then

$$\frac{1}{2i} \int_\gamma \bar{z}\, dz = \int_{\mathbb{C}} \operatorname{ind}_\gamma(z)\, dA(z) = \int_\Omega dA(z) = A(\Omega).$$

□

A consequence of the previous corollary is that under the given assumptions the integral $\displaystyle\int_\gamma \bar{z}\, dz$ is purely imaginary. This can also been seen directly as follows. Suppose $\gamma\colon [a, b] \to \mathbb{C}$ is a parametrization of the piecewise smooth loop γ. Then

$$\begin{aligned}
\operatorname{Re}\left(\int_\gamma \bar{z}\, dz\right) &= \operatorname{Re}\left(\int_a^b \overline{\gamma(t)}\gamma'(t)\, dt\right) \\
&= \frac{1}{2} \int_a^b \left(\overline{\gamma(t)}\gamma'(t) + \gamma(t)\overline{\gamma'(t)}\right) dt \\
&= \frac{1}{2} \int_a^b \frac{d}{dt}\left(\overline{\gamma(t)}\gamma(t)\right) dt = \left[|\gamma(t)|^2\right]_a^b = 0.
\end{aligned}$$

21 Runge's Approximation Theorem

21.1. Background from Approximation Theory. Let $\mathcal{F}$ be a class of functions on a set, say on a compact set $K \subseteq \mathbb{R}^n$ or an open set $U \subseteq \mathbb{R}^n$, and let $\mathcal{S} \subseteq \mathcal{F}$ be a subclass of particularly "nice" or "simple" functions. The basic problem of approximation theory is under what conditions we can approximate functions in $\mathcal{F}$ by functions in $\mathcal{S}$.

An instance for a result in this direction is the *Weierstrass Approximation Theorem*. In this case $\mathcal{F}$ is the set $C(K)$ of all (complex-valued) continuous functions on a compact set $K \subseteq \mathbb{R}^n$, and $\mathcal{S}$ is the set of all polynomials (with complex coefficients) in the standard coordinate functions $x_1, \ldots, x_n$ on $\mathbb{R}^n$. Then according to this theorem for all $f \in \mathcal{F} = C(K)$ and all $\epsilon > 0$ there exists a polynomial $P \in \mathcal{S}$ such that $|f(u) - P(u)| < \epsilon$ for all $u \in K$.

For $n = 1$ and $K = [a, b] \subseteq \mathbb{R}$ the proof can be outlined as follows. If $f \in C([a, b])$, then we may assume that f is a continuous function on a larger interval $[\alpha, \beta] \subseteq \mathbb{R}$ with $[a, b] \subseteq (\alpha, \beta)$. For $l \in \mathbb{N}$ we define

$$P_l(u) = \frac{\displaystyle\int_\alpha^\beta f(x)(1 - (u - x)^2)^l \, dx}{\displaystyle\int_{-1}^1 (1 - x^2)^l \, dx}, \qquad u \in [a, b].$$

Then P_l is a polynomial in u and it not difficult to see that $P_l \to f$ uniformly on K as $l \to \infty$.

In higher dimensions one can implement a proof along very similar lines if one uses the fact that every continuous function on a closed set A in $\mathbb{R}^n$ can be extended to a continuous function on $\mathbb{R}^n$ (one either uses *Tietze's Extension Theorem* to justify this fact or gives a direct simple argument based on $A \subseteq \mathbb{R}^n$).

From the Weierstrass Approximation Theorem one can derive the following consequence. Let $U \subseteq \mathbb{R}^n$ be open, and K_l, $l \in \mathbb{N}$, be a compact exhaustion of U (for this terminology see the proof of Theorem 8.3 and Lemma 21.5 below). Then there exists a polynomial P_l in the standard coordinate functions $x_1, \ldots, x_n$ on $\mathbb{R}^n$ such that $|f(u) - P_l(u)| < 1/l$ for $l \in \mathbb{N}$ and $u \in K_l$. In particular, we can find a sequence of polynomials P_l such that $P_l \to f$ locally uniformly on U as $l \to \infty$.

In $\mathbb{R}^2 \cong \mathbb{C}$ we can replace the standard coordinate functions x and y on $\mathbb{R}^2$ by $z = x + iy$ and $\bar{z} = x - iy$. So every continuous function f on an open set $U \subseteq \mathbb{C}$ is the locally uniform limit of a sequence of polynomials in z and $\bar{z}$. An obvious question here is whether one can strengthen this statement and show that every *holomorphic* function f on U is the locally uniform limit of polynomials in z *alone*.

As the following example shows this is not true in general. Let $U = \mathbb{C} \setminus \{0\}$, and $f(z) = 1/z$. Then $f \in H(U)$, but there is no sequence $\{P_n\}$ of polynomials in z such that $P_n \to f$ locally uniformly on U. Indeed, suppose there was such a sequence, and let $\gamma(t) = e^{it}$ for $t \in [0, 2\pi]$. Then

$$2\pi i = \int_\gamma \frac{dz}{z} = \int_\gamma f(z)\,dz = \lim_{n\to\infty} \underbrace{\int_\gamma P_n(z)\,dz}_{=0} = 0,$$

which is a contradiction.

As we will prove in this section (see Theorem 21.6), one has to impose some restrictions on U to get a statement in this direction, or one has to allow rational functions instead of polynomials.

Lemma 21.2 (Basic approximation lemma)**.** *Let $U \subseteq \mathbb{C}$ be open, $K \subseteq U$ be compact, and $f \in H(U)$. Then for all $\epsilon > 0$ there exists a rational function R of the form*

$$R(w) = \sum_{k=1}^{N} \frac{a_k}{w - z_k},$$

where $N \in \mathbb{N}$, $a_1, \dots, a_N \in \mathbb{C}$, and $z_1, \dots, z_N \in \mathbb{C} \setminus K$, such that

$$|f(w) - R(w)| < \epsilon \quad \text{for } w \in K.$$

Proof. The basic idea for the proof is very simple. We want to represent f on K as a convolution integral of the Cauchy kernel and a function supported away from K, and then approximate the integral by a Riemann sum.

To implement this idea, we pick a compact set K' such that

$$K \subseteq \operatorname{int}(K') \subseteq K' \subseteq U.$$

Then we can find $\varphi \in C_c^\infty(U)$ such that $\varphi(z) - 1 = \varphi_{\bar{z}}(z) = 0$ for $z \in K'$ (Lemma 20.7). Then $u := \varphi f \in C_c^1(\mathbb{C})$ (where it is understood that this function is extended to $\mathbb{C}$ by $u \equiv 0$ outside U). We have $u(w) = f(w)$ for $w \in K$, and $u_{\bar{z}} = \varphi_{\bar{z}} f$ by holomorphicity of f, and so

$$S := \operatorname{supp}(u_{\bar{z}}) \subseteq U \setminus \operatorname{int}(K') \subseteq U \setminus K.$$

Hence $D := \operatorname{dist}(S, K) > 0$.

By Lemma 20.2

$$f(w) = u(w) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{u_{\bar z}(z)}{z-w}\, dA(z) = \frac{1}{\pi}\int_S \frac{u_{\bar z}(z)}{w-z}\, dA(z)$$

for $w \in K$. Now let $\delta \in (0, D/10)$ be small. We can subdivide $\mathbb{C}$ into squares of side-length δ that are *non-overlapping*, i.e., have disjoint interior. Let $Q_1, \dots, Q_N$ be the collection of these squares that meet S, let z_k be the center of Q_k, and

$$a_k := \frac{1}{\pi}\int_{Q_k} u_{\bar z}(z)\, dA(z)$$

for $k = 1\dots, N$. Then $S \subseteq Q_1 \cup \dots \cup Q_N$, and $\operatorname{dist}(K, Q_k) \geq D/2$ for $k = 1, \dots, N$. So for $z \in Q_k$ and $w \in K$ we have

$$\left|\frac{1}{w-z} - \frac{1}{w-z_k}\right| = \frac{|z-z_k|}{|w-z|\cdot|w-z_k|} \leq \frac{\delta}{(D/2)\cdot(D/2)} = \frac{4\delta}{D^2}.$$

Hence for $w \in K$

$$\begin{aligned}\left|f(w) - \sum_{k=1}^N \frac{a_k}{w-z_k}\right| &= \left|\frac{1}{\pi}\sum_{k=1}^N \int_{Q_k}\left(\frac{u_{\bar z}(z)}{w-z} - \frac{u_{\bar z}(z)}{w-z_k}\right) dA(z)\right| \\ &\leq \frac{4\delta}{\pi D^2}\sum_{k=1}^N \int_{Q_k} |u_{\bar z}(z)|\, dA(z) \\ &= \frac{4\delta}{\pi D^2}\underbrace{\int_S |u_{\bar z}(z)|\, dA(z)}_{=:C} = \frac{4C}{\pi D^2}\delta < \epsilon,\end{aligned}$$

if δ is chosen small enough (note that C and D are independent of δ). The claim follows. $\square$

The previous lemma allows us to approximate f on K by a rational function whose poles are outside K. We want to show a stronger statement, where the poles are in a prescribed position depending on K, but independent of f. For this we need the following lemma.

Lemma 21.3 (Pole-Pushing Lemma)**.** *Let $K \subseteq \mathbb{C}$ be compact, V be a component of $\mathbb{C}\setminus K$, and $a \in V$. Define*

$$F_u(w) = \frac{1}{w-u} \quad \textit{for } w \in K \textit{ and } u \in V.$$

Then for each $u \in V$ the function F_u is uniformly approximable *on K by polynomials in F_a, i.e., for each $\epsilon > 0$ there exists $n \in \mathbb{N}$, and $a_0, \dots, a_n \in \mathbb{C}$ such that*

$$\left| \frac{1}{w-u} - \sum_{k=0}^{n} \frac{a_k}{(w-a)^k} \right| < \epsilon \quad \textit{for all } w \in K.$$

If V is the unbounded component of $\mathbb{C} \setminus K$, then for each $u \in V$ the function F_u is uniformly approximable on K by polynomials, i.e., for each $\epsilon > 0$ there exists $n \in \mathbb{N}$, and $a_0, \dots, a_n \in \mathbb{C}$ such that

$$\left| \frac{1}{w-u} - \sum_{k=0}^{n} a_k w^k \right| < \epsilon \quad \textit{for all } w \in K.$$

Proof. The proof consists of several steps. In the following w will always be a variable point in K.

Step 1. For $a \in V$ let $P_a(K) \subseteq C(K)$ be the set of all functions on K that can be written as polynomials in F_a, and $A_a(K) \subseteq C(K)$ be the set of all functions that are uniformly approximable on K by functions in $P_a(K)$ (note that the functions in $A_a(K)$ are continuous as uniform limits of continuous functions). The space $C(K)$ carries a natural metric d induced by the supremum norm on $C(K)$, i.e.,

$$d(g,h) := \sup_{w \in K} |g(w) - h(w)|, \quad g, h \in C(K).$$

Then $f \in A_a(K)$ iff there exists a sequence $\{f_n\}$ in $P_a(K)$ such that $f_n \to f$ uniformly on K iff $d(f_n, f) \to 0$ as $n \to \infty$ iff $f \in \overline{P_a(K)}$. So $A_a(K) = \overline{P_a(K)}$ (where the bar refers to closure in the metric space $(C(K), d)$).

It is clear that $P_a(K)$ is an algebra, i.e., a complex vector space of functions with the additional property that if $f, g \in P_a(K)$, then $fg \in P_a(K)$. Since $A_a(K) = \overline{P_a(K)}$, the space $A_a(K)$ is also an algebra.

Another immediate observation is that a function $f \colon K \to \mathbb{C}$ is uniformly approximable by functions in $A_a(K)$ iff $f \in \overline{A_a(K)} = A_a(K) = \overline{P_a(K)}$ iff f is uniformly approximable by functions in $P_a(K)$.

Step 2. Now suppose that $u \in V$, $r > 0$, $B(u,r) \subseteq V$, $F_u \in A_a(K)$, and $v \in B(u,r)$. We claim that $F_v \in A_a(K)$. By the previous discussion it is enough to show that F_v is uniformly approximable on K by polynomials in F_u (because then F_v is uniformly approximable by functions in the algebra $A_a(K)$, and hence lies in $A_a(K)$ itself).

Now for $w \in K$ we have

$$F_v(w) = \frac{1}{w-v} = \frac{1}{(w-u)-(v-u)}$$
$$= \frac{1}{(w-u)} \cdot \frac{1}{1-\dfrac{v-u}{w-u}} = \sum_{n=0}^{\infty} \frac{(v-u)^n}{(w-u)^{n+1}}. \tag{80}$$

Note that $|w-u| \geq r$ for $w \in K$, and so

$$\left|\frac{v-u}{w-u}\right| \leq \frac{|v-u|}{r} < 1.$$

Hence the convergence of the series in (80) is uniform for $w \in K$ by the Weierstrass M-test. We conclude that F_v is uniformly approximable on K by partial sums of this series and hence by polynomials in F_u as desired.

Step 3. We claim that $F_u \in A_a(K)$ for all $u \in V$. To see this, let $u \in V$ be arbitrary. Since V is open and connected, there exists a polygonal path P in V joining a and u (Theorem 2.31). Then $r := \operatorname{dist}(P, \mathbb{C} \setminus V) > 0$. On P we can find points $u_0 = a, u_1, \ldots, u_n = u$ such that $|u_k - u_{k-1}| < r$, and so $u_k \in B(u_{k-1}, r) \subseteq V$ for all $k = 1, \ldots, n$.

Note that $F_{u_0} = F_a \in P_a(K) \subseteq A_a(K)$, and so $F_{u_1} \in A_a(K)$ by what we have seen in Step 2. Repeating this argument, we conclude that

$$F_{u_0}, F_{u_1}, \ldots, F_{u_n} = F_u \in A_a(K).$$

This establishes the first part of the lemma.

Step 4. Now suppose that V is the unbounded component of K. We denote by $P_\infty(K) \subseteq C(K)$ the set of all functions that can written as polynomials (in w) on K, and by $A_\infty(K) \subseteq C(K)$ the set of all functions that are uniformly approximable on K by functions in $P_\infty(K)$. Again $P_\infty(K)$ and $A_\infty(K)$ are algebras, and $\overline{A_\infty(K)} = A_\infty(K) = \overline{P_\infty(K)}$.

We can pick $a \in V$ such that

$$|a| > R := \sup\{|w| : w \in K\}.$$

Then for $w \in K$ we have

$$F_a(w) = \frac{1}{w-a} = -\frac{1}{a} \cdot \frac{1}{1-w/a} = -\sum_{n=0}^{\infty} \frac{w^n}{a^{n+1}}. \tag{81}$$

Note that $|w/a| \leq R/|a| < 1$, and so the convergence of the series in (81) is uniform on K by the Weierstrass M-test. Hence F_a is uniformly approximable on K by partial sums of this series, and so by elements in $P_\infty(K)$. This shows that $F_a \in A_\infty(K)$. Since $A_\infty(K)$ is an algebra, we conclude that $P_a(K) \subseteq A_\infty(K)$, and so $A_a(K) = \overline{P_a(K)} \subseteq \overline{A_\infty(K)} = A_\infty(K)$.

Now if $u \in V$ is arbitrary, then $F_u \in A_a(K) \subseteq A_\infty(K)$ by what we have seen in Step 2. This shows that F_u is uniformly approximable on K by polynomials as claimed. ☐

Lemma 21.4 (Improved approximation lemma)**.** *Let $U \subseteq \mathbb{C}$ be open, $K \subseteq U$ be compact, and $f \in H(U)$. Suppose $A \subseteq \mathbb{C} \setminus K$ is a set that meets each bounded component of $\mathbb{C} \setminus K$.*

Then for all $\epsilon > 0$ there exists a rational function R that has no poles outside the set $A \cup \{\infty\}$ such that

$$|f(w) - R(w)| < \epsilon \quad \textit{for } w \in K.$$

If $\mathbb{C} \setminus K$ has no bounded component, then for all $\epsilon > 0$ there exists a polynomial P such that

$$|f(w) - P(w)| < \epsilon \quad \textit{for } w \in K.$$

The second part of this lemma can be considered as a special case of the first part if one chooses $A = \emptyset$. Here we consider rational functions as holomorphic maps on $\widehat{\mathbb{C}}$. The non-constant polynomials are precisely those rational functions that have a pole at ∞ and no other poles.

Proof. This immediately follows from Lemma 21.2 if one "pushes" the poles z_k of the approximating rational function R to a location in A. This can be done with arbitrarily small error by Lemma 21.3. If $\mathbb{C} \setminus K$ has no bounded component, then one can push each pole to ∞, i.e., get a polynomial approximation, by the second part of Lemma 21.3. ☐

Lemma 21.5 (Compact exhaustions with good properties)**.** *Let $U \subseteq \mathbb{C}$ be open. Then there exist compact sets $K_n \subseteq U$ for $n \in \mathbb{N}$ such that*

(i) $K_n \subseteq \operatorname{int}(K_{n+1})$ *for* $n \in \mathbb{N}$,

(ii) $U = \bigcup_{n \in \mathbb{N}} K_n$,

(ii) *every bounded component of* $\mathbb{C} \setminus K_n$ *for* $n \in \mathbb{N}$ *contains a bounded component of* $\mathbb{C} \setminus U$.

The first two properties (i) and (ii) just say that the sequence K_n, $n \in \mathbb{N}$, is a compact exhaustion of K_n. The additional property (iii) intuitively says that each "hole" (=bounded complementary component) of a set K_n contains at least one "hole" of U. So the sets K_n may have fewer holes than U, but not additional unnecessary holes.

Proof. For $n \in \mathbb{N}$ we define

$$V_n = \bigcup_{a \in \mathbb{C} \setminus U} B(a, 1/n) \cup \{z \in \mathbb{C} : |z| > n\}, \tag{82}$$

and $K_n = \mathbb{C} \setminus V_n$. Then the sets K_n, $n \in \mathbb{N}$, are compact and have the properties (i) and (ii) (excercise!).

Suppose that $n \in \mathbb{N}$ and C is a bounded component of $\mathbb{C} \setminus K_n = V_n$. Since the sets in the union (82) are connected, it follows that C contains each of these sets that it meets. Since C is also bounded, it follows that

$$C = \bigcup_{a \in A} B(a, 1/n)$$

for some non-empty set $A \subseteq \mathbb{C} \setminus U$. Pick $a \in A$, and let C' be the component of $\mathbb{C} \setminus U$ that contains a. Then $C' \subseteq \mathbb{C} \setminus U \subseteq \mathbb{C} \setminus K_n$ is connected and meets C. Hence $C' \subseteq C$, and so C' is bounded. This shows that C contains a bounded component of $\mathbb{C} \setminus U$. $\square$

Theorem 21.6 (Runge's Approximation Theorem)**.** *Let* $U \subseteq \mathbb{C}$ *be open,* $f \in H(U)$, *and* $A \subseteq \mathbb{C} \setminus U$ *be a set that meets each bounded component of* $\mathbb{C} \setminus U$.

Then there exists a sequence $\{R_n\}$ *of rational functions that have no poles outside the set* $A \cup \{\infty\}$ *such that* $R_n \to f$ *locally uniformly on* U.

If $\mathbb{C} \setminus U$ *has no bounded component, then there exists a sequence* $\{P_n\}$ *of polynomials such that* $P_n \to f$ *locally uniformly on* U.

Again we can consider the second part as a special case of the first part if we choose $A = \emptyset$, but is useful to state this case explicitly.

Proof. Let K_n, $n \in \mathbb{N}$, be a compact exhaustion of U as in Lemma 21.5. Then every bounded component of one of the sets $\mathbb{C} \setminus K_n$ contains a bounded component of $\mathbb{C} \setminus U$, and hence a point in A. By Lemma 21.4 there exists a rational function R_n with no poles outside $A \cup \{\infty\}$ such that

$$|f(w) - R_n(w)| < 1/n \quad \text{for } w \in K_n.$$

Since every compact set $K \subseteq U$ lies in each set K_n for n large enough, it follows that $R_n \to f$ compactly and hence locally uniformly on U.

If $\mathbb{C} \setminus U$ has no bounded component, then none of the sets $\mathbb{C} \setminus K_n$ has a bounded component either. The second part of Lemma 21.4 then shows that we can find sequence $\{P_n\}$ of polynomials such that

$$|f(w) - P_n(w)| < 1/n \quad \text{for } w \in K_n.$$

Again this implies $P_n \to f$ locally uniformly on U as desired. □

Corollary 21.7. *Let $U \subseteq \mathbb{C}$ be a region such that $\widehat{\mathbb{C}} \setminus U$ is connected, and $f \in H(U)$. Then there exists a sequence of polynomials $\{P_n\}$ such that $P_n \to f$ locally uniformly on U.*

Proof. Let K_n, $n \in \mathbb{N}$, be a compact exhaustion of U as in Lemma 21.5. Then for all $n \in \mathbb{N}$ the set $\mathbb{C} \setminus K_n$ has no bounded component.

Indeed, suppose V is a bounded component of one of the sets $\mathbb{C} \setminus K_n$. Then there exists a bounded component C of $\mathbb{C} \setminus U$ such that $C \subseteq V$, and so $(\widehat{\mathbb{C}} \setminus U) \cap V \neq \emptyset$.

Let $V' \subseteq \widehat{\mathbb{C}}$ be the union of all other components of $\mathbb{C} \setminus K_n$ and $\{\infty\}$. Then V and V' are open subsets of $\widehat{\mathbb{C}}$ (Note that $\mathbb{C} \setminus K_n$ has precisely one unbounded component. Together with ∞ it provides an open neighborhood of ∞ that lies in V').

We have $V \cap V' = \emptyset$, and

$$V \cup V' \supseteq (\mathbb{C} \setminus K_n) \cup \{\infty\} = \widehat{\mathbb{C}} \setminus K_n \supseteq \widehat{\mathbb{C}} \setminus U.$$

Since $(\widehat{\mathbb{C}} \setminus U) \cap V \neq \emptyset$ and $\infty \in (\widehat{\mathbb{C}} \setminus U) \cap V' \neq \emptyset$, we get a contradiction to our assumption that $\widehat{\mathbb{C}} \setminus U$ is connected.

Since no set $\mathbb{C} \setminus K_n$ has a bounded component, the improved approximation lemma (Lemma 21.4) provides polynomials P_n such that

$$|f(w) - P_n(w)| < 1/n \quad \text{for } w \in K_n.$$

Then $P_n \to f$ locally uniformly on U. □

References

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