Homework 9 (Due: Mo, 12/02)

Problem 1: Let $\text{Aut}(\mathbb{D})$ be the set of all conformal automorphisms of the unit disk $\mathbb{D}$, i..e., the set of all biholomorphisms $S\colon \mathbb{D} \to \mathbb{D}$.

a) Show that if $S \in \text{Aut}(\mathbb{D})$, then

$$\frac{|S'(z)|}{1-|S(z)|^2} = \frac{1}{1-|z|^2} \quad \text{for} \quad z \in \mathbb{D}.$$

b) Let $\gamma\colon [a,b] \to \mathbb{D}$ be a piecewise smooth path in $\mathbb{D}$. Define its *hyperbolic length* as

$$\ell_h(\gamma) := \int_\gamma \frac{2|dz|}{1-|z|^2} = \int_a^b \frac{2|\gamma'(t)|}{1-|\gamma(t)|^2}\, dt.$$

Show that if $S \in \text{Aut}(\mathbb{D})$, then

$$\ell_h(S \circ \gamma) = \ell_h(\gamma).$$

c) Define the *hyperbolic distance* between two points $z, w \in \mathbb{D}$ as

$$d_h(z,w) := \inf_\gamma \ell_h(\gamma),$$

where the infimum is taken over all piecewise smooth paths γ in $\mathbb{D}$ with endpoints z and w. Show that d_h is a metric on $\mathbb{D}$, and that each $S \in \text{Aut}(\mathbb{D})$ is an isometry on the space $(\mathbb{D}, d_h)$, i.e., we have

$$d_h(S(z), S(w)) = d_h(z,w) \quad \text{for all} \quad z, w \in \mathbb{D}.$$

Problem 2:

a) Show that a circle C intersects the unit circle $\partial\mathbb{D}$ orthogonally if and only if either C is a line passing through 0 or $C = \{z \in \mathbb{C} : |z-a| = r\}$, where $a \in \mathbb{C}$, $|a| > 1$, and $r = \sqrt{|a|^2 - 1}$.

b) Suppose $z, w \in \mathbb{D}$ are distinct points. Show that there exists a unique map $S \in \text{Aut}(\mathbb{D})$ such that $S(z) = 0$ and $S(w) \in (0,1)$. Express $S(w)$ in terms of z and w.

c) Show that there exists a unique circle C that passes through z and w and intersects the unit circle $\partial\mathbb{D}$ orthogonally.

Problem 3: Let $r \in (0,1)$ and $\gamma\colon [0,1] \to \mathbb{D}$ be a piecewise smooth path in $\mathbb{D}$ with $\gamma(0) = r$ and $\gamma(1) = 0$.

a) Define
$$s(t) := \int_0^t |\gamma'(u)|\, du \quad \text{for} \quad t \in [0,1].$$
Show that
$$|\gamma(t)| \geq \max\{0, r - s(t)\} \quad \text{for} \quad t \in [0,1].$$

b) Show that
$$\ell_h(\gamma) \geq \int_0^r \frac{2ds}{1-(r-s)^2}.$$

c) Show that the hyperbolic distance between 0 and r is equal to the hyperbolic length of the line segment joining 0 and r. Compute this length explicitly.

Problem 4: Let $z, w \in \mathbb{D}$ be two distinct points, C the unique circle that passes through z and w and intersects the unit circle $\partial\mathbb{D}$ orthogonally, and A the unique subarc of C contained in $\mathbb{D}$ with endpoints z and w.

a) Find an explicit expression for the hyperbolic distance $d_h(z,w)$.

b) Show that A is a *hyperbolic geodesic segment* connecting z and w, i.e., a path in $\mathbb{D}$ with endpoints z and w whose hyperbolic length is equal to the hyperbolic distance of z and w.

c) The circle C intersects $\partial\mathbb{D}$ in two points u, v. Suppose the notation is chosen such that the points z, w, u, v are in cyclic order on C. Show that
$$d_h(z,w) = \log(z,w,u,v).$$