Homework 8 (Due: Fr, 11/22)

Problem 1: Let X be a topological 2-manifold, and $\mathcal{A} = \{\phi_i : U_i \to V_i : i \in I\}$ be a complex atlas on X, i.e., an atlas of holomorphically compatible charts.

a) Suppose that $\psi_1 : M_1 \to N_1$ and $\psi_2 \colon M_2 \to N_2$ are two charts on X that are holomorphically compatible with each chart in $\mathcal{A}$. Show that ψ_1 and ψ_2 are holomorphically compatible.

b) Let $\mathcal{B}$ be the set of all charts on X that are holomorphically compatible with each chart in $\mathcal{A}$. Show that $\mathcal{B}$ is a maximal complex atlas containing $\mathcal{A}$, i.e., $\mathcal{B}$ is a complex atlas containing $\mathcal{A}$, and every chart that is holomorphically compatible with all the charts in $\mathcal{A}$ is contained in $\mathcal{B}$.

c) Suppose that $\mathcal{A}'$ is another complex atlas on X, and let $\mathcal{B}'$ be the associated maximal complex atlas as defined in b). Show that $\mathcal{A}$ and $\mathcal{A}'$ are analytically equivalent if and only if $\mathcal{B} = \mathcal{B}'$.

Problem 2: Let X and Y be Riemann surfaces, and $f \colon X \to Y$ be a continuous map. Recall that f is called *holomorphic* if for each complex chart $\varphi \colon U_1 \to V_1$ on X and each complex chart $\psi \colon U_2 \to V_2$ on Y with $\varphi(U_1) \subseteq U_2$ the function $\psi \circ f \circ \varphi^{-1} \colon V_1 \to V_2 \subseteq \mathbb{C}$ is holomorphic on the open set $V_1 \subseteq \mathbb{C}$.

Show that this is equivalent to the following condition: for each $x \in X$ there *exists* a complex chart $\varphi \colon U_1 \to V_1$ on X, and a complex chart $\psi \colon U_2 \to V_2$ on Y with $x \in U_1$ and $\varphi(U_1) \subseteq U_2$ such that $\psi \circ f \circ \varphi^{-1} \colon V_1 \to V_2$ is holomorphic on V_1.

Problem 3: Let X and Y be connected Riemann surfaces, and $f \colon X \to Y$ and $g \colon X \to Y$ be holomorphic maps. Suppose that the set

$$A = \{x \in X : f(x) = g(x)\}$$

has a limit point in X. Show that $f = g$.

Problem 4: Let $\Omega \subseteq \mathbb{C}$ be a region, $n \in \mathbb{N}$, and $a_0, \dots, a_{n-1} \in H(\Omega)$.

a) Show that for each $z_0 \in \Omega$ there exists a constant $r_0 > 0$ with the following property: if f is any holomorphic germ at z_0 (cf. HW 7, Prob. 4) that satisfies the differential equation

$$f^{(n)} + a_{n-1} f^{(n-1)} + \cdots + a_0 f = 0 \qquad (1)$$

near z_0, then the power series representing f has radius of convergence $R \geq r_0$.

b) Let f be a holomorphic germ at a point $z_0 \in \Omega$ and suppose that f satisfies the equation (1) locally near z_0. Show that if $\gamma\colon [0,1] \to \Omega$ is a path in Ω, then f has an analytic continuation along γ (cf. HW 7, Prob. 4) and that this analytic continuation produces a holomorphic germ g at $z_1 := \gamma(1)$ that satisfies the differential equation (1) (for $f = g$) locally near z_1.