Homework 7 (Due: Fr, 11/15)

Problem 1: Let $f\colon \overline{\mathbb{D}} \to \mathbb{C}$ be a continuous function such that $f|_{\mathbb{D}}$ is holomorphic on $\mathbb{D}$. Suppose that $|f(e^{i\theta})| \leq 1$ for $\theta \in [0, \pi)$ and $|f(e^{i\theta})| \leq 1/4$ for $\theta \in [\pi, 2\pi)$. Show that $|f(0)| \leq 1/2$.

Problem 2: A function $B\colon \mathbb{D} \to \mathbb{C}$ is called a (finite) Blaschke product if there exist $\theta \in \mathbb{R}$ and $a_1, \dots a_n \in \mathbb{D}$ such that

$$B(z) = e^{i\theta} \prod_{k=1}^{n} \frac{z - a_k}{1 - \overline{a}_k z}$$

for $z \in \mathbb{D}$.

a) Show that every Blaschke product B is continuous on $\overline{\mathbb{D}}$, holomorphic on $\mathbb{D}$, and satisfies $|B(z)| = 1$ for $|z| = 1$.

b) Show that conversely every continuous function $f\colon \overline{\mathbb{D}} \to \mathbb{C}$ that is holomorphic on $\mathbb{D}$ and satisfies $|f(z)| = 1$ for $|z| = 1$ is a Blaschke product.

Problem 3: Let $\Omega \subseteq \mathbb{C}$ be a convex set and $f \in H(\Omega)$. Show that if $\mathrm{Re} f' > 0$ on Ω, then f is injective.

Problem 4: A *holomorphic germ* P at a point $z_0 \in \mathbb{C}$ is a power series $P(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ centered at z_0 with positive radius of convergence. Suppose we have a (continuous) path $\gamma\colon [0, 1] \to \mathbb{C}$ and germs P_t at $z_t = \gamma(t)$ for each $t \in [0, 1]$. We say that this family P_t, $t \in [0, 1]$, of germs is *compatible along* γ if for each $t_0 \in [0, 1]$ there exist a holomorphic function f in a neighborhood of z_{t_0} such that P_t is the power series expansion of f at z_t for all t sufficiently close to t_0.

Let $z_0, z_1 \in \mathbb{C}$, $\gamma\colon [0, 1] \to \mathbb{C}$ be a path in $\mathbb{C}$ with $\gamma(0) = z_0$ and $\gamma(1) = z_1$, and Q_0 and Q_1 be germs at z_0 and z_1, respectively. Then Q_1 is called the *analytic continuation* of Q_0 along γ if there exits a family of germs P_t, $t \in [0, 1]$, that is compatible along γ with $P_0 = Q_0$ and $P_1 = Q_1$.

Show that if the analytic continuation Q_1 of a germ Q_0 along a given path γ exists, then it is uniquely determined.