Homework 6 (Due: Mo 11/11)

Problem 1: Let $\Omega \subseteq \mathbb{C}$ be a region, $n \in \mathbb{N}$, and $a_0, \ldots, a_{n-1} \in H(\Omega)$.

a) Suppose that $f \in H(\Omega)$ satisfies the differential equation

$$f^{(n)} + a_{n-1} f^{(n-1)} + \cdots + a_0 f = 0. \tag{1}$$

Show that if there exists a point $z_0 \in \Omega$ such that

$$f(z_0) = f'(z_0) = \cdots = f^{(n-1)}(z_0) = 0,$$

then f vanishes identically on Ω.

b) Let $z_0 \in \Omega$. Show that for given values $f(z_0)$, $f'(z_0), \ldots, f^{(n-1)}(z_0)$ the equation (1) has a unique solution f that is holomorphic near z_0. Hint: Assume that

$$f(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k.$$

Show that the coefficients c_k can be computed recursively. To establish convergence of the series near z_0, prove an estimate of the form $|c_k| \leq CM^k$ by induction.

Problem 2: Let $f \colon \mathbb{C} \to \mathbb{C}$ be an entire function and suppose that $f(\sqrt[n]{n}) \in \mathbb{R}$ for all $n \in \mathbb{N}$. Show that $f(\mathbb{R}) \subseteq \mathbb{R}$.

Problem 3: Let $\Omega \subseteq \mathbb{C}$ be open, and $f \in H(\Omega)$. Show that for every point $z_0 \in \Omega$ there exist distinct points $z_1, z_2 \in \Omega$ such that

$$f'(z_0) = \frac{f(z_1) - f(z_2)}{z_1 - z_2}.$$

Problem 4: Suppose that the function f is holomorphic on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and injective on $\mathbb{D} \setminus \{0\}$. Show that f is injective on $\mathbb{D}$.