Problem 1: Let $\Omega \subseteq \mathbb{C}$ be a convex region, and $f$ a zero-free holomorphic function on $\Omega$.

a) Show that $f$ has a holomorphic logarithm, i.e., there exists a function $L \in H(\Omega)$ such that
$$f(z) = \exp(L(z)) \quad \text{for all} \quad z \in \Omega.$$  

Hint: Do not use the function $w \mapsto \log(w)$ for $w \in \mathbb{C} \setminus \{0\}$ which we have not yet defined (and cannot be defined as a holomorphic function on $\mathbb{C} \setminus \{0\}$).

b) Show that $f$ has a holomorphic square root, i.e., there exists a function $S \in H(\Omega)$ such that
$$f(z) = S(z)^2 \quad \text{for all} \quad z \in \Omega.$$

Problem 2:

a) Let $f$ be a continuous function in $\mathbb{C}$, and $\{z_n\}$ and $\{w_n\}$ be convergent sequences in $\mathbb{C}$ with $z_n \to z$ and $w_n \to w$ as $n \to \infty$. Show that
$$\lim_{n \to \infty} \int_{[z_n,w_n]} f(z) \, dz = \int_{[z,w]} f(z) \, dz.$$  

b) Let $f$ be a continuous function in $\mathbb{C}$ that is holomorphic on $\mathbb{C} \setminus \mathbb{R}$. Show that $f$ is holomorphic on $\mathbb{C}$.

Problem 3:

a) Show that the power series
$$\sum_{n=0}^{\infty} \frac{i^{n+1}}{(n+1)!} z^n$$

converges for all $z \in \mathbb{C}$.

b) Let $g$ be the function represented by the power series in a). Find a simple expression for $g(z)$ if $z \neq 0$.

c) For $r > 0$ let $\alpha_r : [0, 2] \to \mathbb{C}$ be the path defined by
$$\alpha_r(t) = \begin{cases} 
  r e^{i \pi t}, & t \in [0, 1], \\
  (2t - 3)r, & t \in [1, 2].
\end{cases}$$

Show that $\int_{\alpha_r} g(z) \, dz = 0$. 
d) Show that
\[ \lim_{r \to \infty} \int_{[-r,r]} g(z) \, dz = i\pi. \]
e) Show that
\[ \int_{0}^{\infty} \frac{\sin x}{x} \, dx = \frac{\pi}{2}. \]

**Problem 4:** Let \( f \) be a holomorphic function in the unit disk \( \mathbb{D} \). Suppose that \( |f(1/n)| \leq e^{-n} \) for all \( n \in \mathbb{N}, \ n \geq 2 \). Show that \( f \) vanishes identically.