Homework 5 (Due: Fr 11/01)

Problem 1: Let $\Omega \subseteq \mathbb{C}$ be a convex region, and f a zero-free holomorphic function on Ω.

a) Show that f has a holomorphic logarithm, i.e., there exists a function $L \in H(\Omega)$ such that
$$f(z) = \exp(L(z)) \quad \text{for all} \quad z \in \Omega.$$
Hint: Do not use the function $w \mapsto \log(w)$ for $w \in \mathbb{C} \setminus \{0\}$ which we have not yet defined (and cannot be defined as a holomorphic function on $\mathbb{C} \setminus \{0\}$).

b) Show that f has a holomorphic square root, i.e., there exists a function $S \in H(\Omega)$ such that
$$f(z) = S(z)^2 \quad \text{for all} \quad z \in \Omega.$$

Problem 2:

a) Let f be a continuous function in $\mathbb{C}$, and $\{z_n\}$ and $\{w_n\}$ be convergent sequences in $\mathbb{C}$ with $z_n \to z$ and $w_n \to w$ as $n \to \infty$. Show that
$$\lim_{n\to\infty} \int_{[z_n, w_n]} f(z)\, dz = \int_{[z,w]} f(z)\, dz.$$

b) Let f be a continuous function in $\mathbb{C}$ that is holomorphic on $\mathbb{C} \setminus \mathbb{R}$. Show that f is holomorphic on $\mathbb{C}$.

Problem 3:

a) Show that the power series
$$\sum_{n=0}^{\infty} \frac{i^{n+1}}{(n+1)!} z^n$$
converges for all $z \in \mathbb{C}$.

b) Let g be the function represented by the power series in a). Find a simple expression for $g(z)$ if $z \neq 0$.

c) For $r > 0$ let $\alpha_r \colon [0,2] \to \mathbb{C}$ be the path defined by
$$\alpha_r(t) = \begin{cases} re^{i\pi t}, & t \in [0,1], \\ (2t-3)r, & t \in [1,2]. \end{cases}$$
Show that $\displaystyle\int_{\alpha_r} g(z)\, dz = 0.$

d) Show that

$$\lim_{r\to\infty} \int_{[-r,r]} g(z)\,dz = i\pi.$$

e) Show that

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

Problem 4: Let f be a holomorphic function in the unit disk $\mathbb{D}$. Suppose that $|f(1/n)| \le e^{-n}$ for all $n \in \mathbb{N}$, $n \ge 2$. Show that f vanishes identically.