Homework 4 (Due: 10/25)

Problem 1:

a) Suppose $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is a power series with nonzero coefficients. Show that its radius of convergence R is given by

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$

if the limit exists.

b) Find the radius of convergence of the power series $\sum_{n=1}^{\infty} a_n z^n$, where

(i) $a_n = (2i)^n/(n+i)^2$,
(ii) $a_n = n^2 + 5in + 1$,
(iii) $a_n = n^{\log n}$.

Problem 2:

a) Suppose f and g are holomorphic functions. Show that for all $n \in \mathbb{N}$ the so-called *Leibniz formula* is valid:

$$(f\cdot g)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} \cdot g^{(n-k)}.$$

b) Suppose that the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ have radii of convergence $R_1 > 0$ and $R_2 > 0$. For $n \in \mathbb{N}_0$ define

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Show that the power series $h(z) = \sum_{n=0}^{\infty} c_n z^n$ (the so-called *Cauchy product* of f and g) converges for all $|z| < R := \min\{R_1, R_2\}$ and that $h(z) = f(z)g(z)$ for all such z.

Problem 3: Let $R > 0$, and suppose that the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converges for all $z \in B := B(0, R)$.

a) Show that the power series converges uniformly on each compact subset of B.
b) For $n \in \mathbb{N}_0$ and $z \in B(0, R)$ define

$$S_n(z) := \left| \sum_{k=0}^{n} a_k z^k \right|^2.$$

Show that on each compact subset of B the function sequence $\{S_n\}$ converges uniformly to $|f|^2$.
c) Show that for each $0 \leq r < R$ *Parseval's formula* is valid:

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 \, dt = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

Problem 4: Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges for $z \in \mathbb{C}$ if and only if $|z| \leq 1$ and $z \neq 1$.