Homework 4 (Due: 10/25)

Problem 1:

a) Suppose \( \sum_{n=0}^{\infty} a_n (z-z_0)^n \) is a power series with nonzero coefficients. Show that its radius of convergence \( R \) is given by

\[
R = \lim_{n \to \infty} \frac{|a_n|}{|a_{n+1}|}
\]

if the limit exists.

b) Find the radius of convergence of the power series \( \sum_{n=1}^{\infty} a_n z^n \), where

(i) \( a_n = (2i)^n/(n + i)^2 \),
(ii) \( a_n = n^2 + 5in + 1 \),
(iii) \( a_n = n \log n \).

Problem 2:

a) Suppose \( f \) and \( g \) are holomorphic functions. Show that for all \( n \in \mathbb{N} \) the so-called Leibniz formula is valid:

\[
(f \cdot g)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} \cdot g^{(n-k)}.
\]

b) Suppose that the power series \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( g(z) = \sum_{n=0}^{\infty} b_n z^n \) have radii of convergence \( R_1 > 0 \) and \( R_2 > 0 \). For \( n \in \mathbb{N}_0 \) define

\[
c_n = \sum_{k=0}^{n} a_k b_{n-k}.
\]

Show that the power series \( h(z) = \sum_{n=0}^{\infty} c_n z^n \) (the so-called Cauchy product of \( f \) and \( g \)) converges for all \( |z| < R := \min\{R_1, R_2\} \) and that \( h(z) = f(z)g(z) \) for all such \( z \).

Problem 3: Let \( R > 0 \), and suppose that the power series \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) converges for all \( z \in B := B(0, R) \).
a) Show that the power series converges uniformly on each compact subset of $B$.

b) For $n \in \mathbb{N}_0$ and $z \in B(0, R)$ define

$$S_n(z) := \left| \sum_{k=0}^{n} a_k z^k \right|^2.$$

Show that on each compact subset of $B$ the function sequence $\{S_n\}$ converges uniformly to $|f|^2$.

c) Show that for each $0 \leq r < R$ Parseval’s formula is valid:

$$\frac{1}{2\pi} \int_{0}^{2\pi} |f(re^{it})|^2 \, dt = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

**Problem 4:** Show that the series $\sum_{n=1}^{\infty} z^n$ converges for $z \in \mathbb{C}$ if and only if $|z| \leq 1$ and $z \neq 1$. 