Homework 2 (due: Fr, 10/11)

Problem 1: Let (X, d) be a metric space. Recall that a set $M \subseteq X$ is called *connected* if the only sets $W \subseteq M$ that are both relatively open and relatively closed in M are $W = \emptyset$ and $W = M$.

Show that $M \subseteq X$ is connected if and only if the following condition is true: whenever $U, V \subseteq X$ are open, $M \subseteq U \cup V$, and $U \cap V = \emptyset$, then $M \subseteq U$ or $M \subseteq V$.

Problem 2:

a) Let (X, d) be a metric space, I be an index set, and $M_i \subseteq X$ be connected for $i \in I$. Show that if $\bigcap_{i \in I} M_i \neq \emptyset$, then $\bigcup_{i \in I} M_i$ is connected.

b) Show that if $\Omega \subseteq \mathbb{C}$ is open, and every two points $u, v \in \Omega$ can be joined by a polygonal path in Ω, then Ω is a region (this is a converse of a theorem proved in class).

Problem 3: Let M be a non-empty subset of $\mathbb{C}$. A *(connected) component* C of M is a maximal (with respect to inclusion) connected subset of M. Show that:

a) Each component C of M is relatively closed in M.

b) If C and C' are distinct components of M, then $C \cap C' = \emptyset$.

c) Every connected subset A of M lies in a unique component C of M.

d) M is the (disjoint) union of its components.

e) There exists an equivalence relation $\sim$ on M whose equivalence classes are the components of M. Give a concise description of $\sim$ that does not use the concept of a component!

Problem 4: Suppose K is a non-empty compact subset of $\mathbb{C}$.

a) Let C be a component of K (cf. Prob. 3). Show that C is equal to the intersection of all sets A with $C \subseteq A \subseteq K$ that are relatively open and closed in K.

b) Let C and C' be two distinct components of K. Show that there exist open sets $U, U' \subseteq \mathbb{C}$ such that $C \subseteq U$, $C' \subseteq U'$, and $U \cap U' = \emptyset$.