Homework 1 (due: Mo, 10/7)

Problem 1:

a) Let (X, d) be a metric space, and $A \subseteq X$ be a connected set. Show that every set $B \subseteq X$ with $A \subseteq B \subseteq \overline{A}$ is also connected.

b) Suppose $f\colon \mathbb{C} \to \mathbb{C}$ is a continuous function with $\exp(f(z)) = 1$ for all $z \in \mathbb{C}$. Show that f is constant.

Problem 2:

a) Show that the function $z \mapsto \frac{z^2+3}{z+1}$ is continuous on $\mathbb{C} \setminus \{-1\}$ by using the ϵ-δ-definition of continuity.

b) Suppose $\{z_n\}$ and $\{w_n\}$ are convergent sequences in $\mathbb{C}$, say $z_n \to z$ and $w_n \to w$. Assume $w_n \neq 0$ for $n \in \mathbb{N}$ and $w \neq 0$. Show that $z_n/w_n \to z/w$ by using the ϵ-definition for convergence.

Problem 3:

a) Suppose that $z, w \in \mathbb{C}$, $|z|, |w| < 1$. Show that $\left| \dfrac{z-w}{1-\bar{z}w} \right| < 1$.

b) The set $M = \{z \in \mathbb{C} : |z-1|/|z+2i| < \sqrt{2}\}$ is the complement of a closed disk. Determine its center and its radius.

Problem 4:

a) Let $P(z) = a_0 + a_1 z + \cdots + a_{k-1} z^{k-1} + z^k$ be a polynomial of degree $k \in \mathbb{N}$ with complex coefficients $a_0, \ldots, a_{k-1}$. Suppose that $P(z) = \prod_{j=1}^{k} (z - z_j)$. Find an expression for $\sum_{j=1}^{k} z_j^2$ in terms of the coefficients of P.

b) Let $n \in \mathbb{N}$ and $\zeta_k = \exp\left(\frac{2\pi i k}{n}\right)$ for $k = 1, \ldots, n-1$. Find a simple expression for

$$\sum_{k=1}^{n-1} \frac{1}{(\zeta_k - 1)^2}$$

in terms of n.