**Homework 3** (due: Mo, April 30)

**Problem 1:** For $n \in \mathbb{N}_0$ and $t \in \mathbb{R}$ we consider the Dirichlet kernel
\[ D_n(t) = \frac{\sin((n + 1)t/2)}{\sin(t/2)} \]
and the Fejér kernel
\[ K_n(t) = \frac{1}{n+1} (D_0(t) + \cdots + D_n(t)) = \frac{1}{n+1} \left( \frac{\sin((n + 1)t/2)}{\sin(t/2)} \right)^2. \]

a) Show that the kernels $K_n$ have the following properties:

(i) $K_n$ for $n \in \mathbb{N}_0$ is a non-negative, $2\pi$-periodic, and measurable function on $\mathbb{R}$ with \( \frac{1}{2\pi} \int_{[-\pi,\pi]} K_n(t) \, dt = 1. \)

(ii) for each $\delta > 0$ we have \( \lim_{n \to \infty} \int_{[-\pi,\pi] \setminus [-\delta,\delta]} K_n(t) \, dt = 0. \)

b) Show that a sequence $P_n$, $n \in \mathbb{N}_0$, of kernels with the properties (i) and (ii) as in (a) forms an approximate identity on $\mathbb{T}$ in the following sense: for each $f \in C(\mathbb{T})$ we have
\[ (P_n * f)(x) := \frac{1}{2\pi} \int_{-\pi}^{\pi} P_n(x-t)f(t) \, dt \to f(x) \text{ as } n \to \infty \]
uniformly for $x \in \mathbb{R}$.

c) Let $s_n f$ denote the $n$-th partial sum of the Fourier series of a function $f \in L^1(\mathbb{T})$ and consider
\[ \sigma_n f = \frac{1}{n+1} (s_0 f + \cdots + s_n f). \]
Show that if $f \in C(\mathbb{T})$, then
\[ \|\sigma_n f - f\|_\infty \to 0 \text{ as } n \to \infty. \]

**Problem 2:** Let $f(t) = \sum c_n e^{int}$ be a trigonometric polynomial. Then its (discrete) Hilbert transform $Hf$ is defined as
\[ Hf(t) = -i \sum_{n \leq -1} c_n e^{int} + i \sum_{n \geq 1} c_n e^{int}. \]
One can show that the Hilbert transform is bounded on $L^p(\mathbb{T})$ for $1 < p < \infty$. More precisely, there exists a constant $C_p \geq 0$ such that

$$\|Hf\|_p \leq C_p\|f\|_p$$

for each trigonometric polynomial $f$ (this is a rather difficult theorem that you can use without further justification in this problem).

a) Show that for $1 < p < \infty$ the Hilbert transform extends uniquely to a bounded linear operator $H: L^p(\mathbb{T}) \to L^p(\mathbb{T})$.

b) For $n \in \mathbb{N}$ let $s_n f$ be the $n$-th partial sum of the Fourier series of the trigonometric polynomial $f$. Show that one can represent $s_n f$ as a sum of four terms involving the operator $H$ and two Fourier coefficients of $f$.

Hint: One of the terms is

$$\frac{1}{2i}u_n H(f u_{-n}),$$

where $u_{\pm n}(t) = e^{\pm int}$.

c) Use the previous facts to show that for each $1 < p < \infty$ there exists a constant $C'_p \geq 0$ such that

$$\|s_n f\|_p \leq C'_p\|f\|_p$$

for all $n \in \mathbb{N}_0$ and $f \in L^p(\mathbb{T})$. In other words, the operators $f \mapsto s_n f$ have uniformly bounded operator norms on $L^p(\mathbb{T})$.

d) Use (c) to show that if $f \in L^p(\mathbb{T})$ with $1 < p < \infty$, then the Fourier series of $f$ converges to $f$ in $L^p(\mathbb{T})$, or equivalently,

$$\|s_n f - f\|_p \to 0 \text{ as } n \to \infty.$$ 

**Problem 3:** A function $f: \mathbb{R}^n \to \mathbb{C}$ is called a Schwartz function if it is $C^\infty$-smooth and if all of its partial derivatives $\partial^\alpha f(x)$ tend to 0 as $|x| \to \infty$ faster than any polynomial rate; more precisely, we require that for each multi-index $\alpha$ and each $N \in \mathbb{N}_0$ we have

$$\partial^\alpha f(x) = o((1 + |x|)^{-N}) \text{ as } |x| \to \infty.$$ 

a) Show that for $f \in C^\infty(\mathbb{R}^n)$ the last condition is equivalent to the following condition: for each multi-index $\alpha$ and each $N \in \mathbb{N}_0$ there exists a constant $C = C(\alpha, N) \geq 0$ such that

$$(1 + |x|)^N |\partial^\alpha f(x)| \leq C \text{ for all } x \in \mathbb{R}^n.$$ 

b) Show that the function $x \in \mathbb{R}^n \mapsto e^{-|x|^2}$ is a Schwartz function.

c) Show that if $f$ and $g$ are Schwartz functions on $\mathbb{R}^n$, then $(g * f)(x)$ is defined for each $x \in \mathbb{R}^n$ and $f * g$ is also a Schwartz function.
**Problem 4:** Let $f \in C(T)$ and suppose that $\hat{f}(n) \geq 0$ for each $n \in \mathbb{Z}$. Show that then $\sum_{n \in \mathbb{Z}} \hat{f}(n) < \infty$. 