Exercises

Problem 1: a) Let \([a, b] \subseteq \mathbb{R}\) be a compact interval and \(f, g: [a, b] \to \mathbb{C}\) be absolutely continuous functions.
(b) Prove the following integration-by-parts formula:
\[
\int_a^b f'(x)g(x) \, dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x) \, dx.
\]
Justify the existence of all derivatives and integrals involved!

Problem 2: Show that if \(f \in L^1_{\text{loc}}(\mathbb{R})\), then there exists \(g \in L^1_{\text{loc}}(\mathbb{R})\) such that \(g' = f\) in the sense of distributions.

Problem 3: Show that the following statement false in general: if \(f, g \in L^1_{\text{loc}}(\mathbb{R})\) and \(f' = g\) in the distributional sense, then \(f'(x)\) exists in the classical sense and \(f'(x) = g(x)\) for almost every \(x \in \mathbb{R}\).

Problem 4: Let \(s_k \in \mathcal{S}(\mathbb{R}^n)\) for \(k \in \mathbb{N} \cup \{\infty\}\) and suppose that \(s_k \to s_\infty\) as \(k \to \infty\) in the topology of \(\mathcal{S}(\mathbb{R}^n)\). Show that then for each multi-index \(\alpha\) and for each \(1 \leq p \leq \infty\) we have \(\|\partial^\alpha s_k - \partial^\alpha s_\infty\|_p \to 0\) as \(k \to \infty\) (here \(\|\cdot\|_p\) denotes the \(L^p\)-norm).

Problem 5: Let \(f \in L^1_{\text{loc}}(\mathbb{R}^n)\) and suppose that \(\partial_k f = 0\) in the distributional sense for all \(k = 1, \ldots, n\). Show that then there exists a constant \(c \in \mathbb{C}\) such that \(f = c\) a.e. on \(\mathbb{R}\). Hint: Consider a mollification \(\varphi \ast f\).

Problem 6: (a) Let \(a \in \mathbb{R}^n, \ R > 0\), and consider the sphere
\[
\Sigma(a, R) = \{x \in \mathbb{R}^n : |x - a| = R\}
\]
of radius \(R\) centered at \(a\). Then \(\Sigma(a, R)\) carries a natural Borel measure \(\sigma\) uniquely determined by the relation
\[
\sigma(B) = n\lambda_n(\widetilde{B})
\]
for each Borel set \(B \subseteq \Sigma(a, R)\). Here \(\widetilde{B}\) is the cone with tip \(a\) and base \(B\), i.e,
\[
\widetilde{B} = \{a + t(b - a) : b \in B, \ t \in (0, 1)\}.
\]
Note that this is very similar to how we defined spherical measure on the unit sphere \(\Sigma(0, 1)\).
Let \(f: \Sigma(a, R) \to [0, \infty]\) be a Borel function.
(a) Show that
\[ \int_{\Sigma(a,R)} f \, d\sigma = R^{n-1} \int_{\Sigma(0,1)} f(a + R\xi) \, d\sigma(\xi). \]

(b) Show that
\[ \int_{\Sigma(a,R)} f \, d\sigma = R^{n-1} \int_{B_{n-1}} \frac{1}{y_n} \left[ f(a + R(y,y_n)) - f(a + R(y,-y_n)) \right] \, d\lambda_{n-1}(y). \]

Here \( B_{n-1} \) is the open unit ball in \( \mathbb{R}^{n-1} \) and we set \( y_n = \sqrt{1 - |y|^2} \) for \( y \in B_{n-1} \).

**Problem 7:** We use the notation from Problem 6. We denote by
\[ n(x) = \frac{1}{|x - a|} (x - a) \]
the unit normal vector to the hypersurface \( \Sigma(a,R) \) at a point \( x \in \Sigma(a,R) = \partial B(a,R) \) pointing “outward”. Let \( v = (v_1, \ldots, v_n) \) be a \( C^1 \)-smooth vector field (i.e., an \( \mathbb{R}^n \)-valued function) defined in an open neighborhood of \( \overline{B}(a,R) \).

(a) Show that then
\[ \int_{\overline{B}(a,R)} \text{div} \, v \, d\lambda_n = \int_{\Sigma(a,R)} v \cdot n \, d\sigma. \]

Here
\[ \text{div} \, v = \frac{\partial v_1}{\partial x_1} + \cdots + \frac{\partial v_n}{\partial x_n} \]
with the standard coordinate functions \( x_1, \ldots, x_n \). This is a special case of Gauss’s Theorem. Hint: First consider vector fields of the form \( v = (0, \ldots, 0, v_n) \) and use Problem 6.

(b) Prove the following more general version of Gauss’s Theorem. Consider the an open set \( \Omega \subseteq \mathbb{R}^n \) of the form \( \Omega = B_0 \setminus (B_1 \cup \cdots \cup B_k) \), where \( B_0 \) is an open ball in \( \mathbb{R}^n \) and \( B_1, \ldots, B_k \) are pairwise disjoint closed balls in \( \mathbb{R}^n \) contained in \( B_0 \). Suppose \( v \) is a \( C^1 \)-smooth vector field defined in an open neighborhood of \( \overline{\Omega} \). Then
\[ \int_{\Omega} \text{div} \, v \, d\lambda_n = \int_{\partial \Omega} v \cdot n \, d\sigma. \]

Here \( n \) again denotes the “outward” normal unit vector defined on
\[ \partial \Omega = \partial B_0 \cup \cdots \cup \partial B_k. \]

So on \( \partial B_0 \) it is equal to the surface normal defined in (a), but on \( \partial B_1, \ldots, \partial B_k \) it differs by a sign.

**Problem 8:** (a) Let \( \Omega \subseteq \mathbb{R}^n \) be an open set as in Problem 7, and \( f, g \) be \( C^2 \)-smooth functions defined in an open neighborhood of \( \overline{\Omega} \). Use Gauss’s Theorem to
prove the *Gauss-Green formula*:

\[
\int_{\Omega} (f \Delta g - g \Delta f) \, d\lambda_n = \int_{\partial \Omega} \left( f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n} \right) \, d\sigma.
\]

Here

\[
\frac{\partial h}{\partial n} = \nabla h \cdot n
\]
denotes the derivative of the function \( h \) in the direction of the outward normal \( n \) at a point \( x \in \partial \Omega \).

(b) Use the Gauss-Green formula to give an alternative proof of the fact that on \( \mathbb{R}^2 \) we have

\[
\Delta_x \log |x| = 2\pi \delta_0
\]
in the distributional sense.

**Problem 9:** Find a *fundamental solution* of the Laplace equation in all dimensions \( n \in \mathbb{N} \), i.e., a distribution \( T \) on \( \mathbb{R}^n \) such that

\[
\Delta T = \delta_0.
\]