Homework 2 (due: Fr, Jan. 19)

Problem 1: Let \((X, \mathcal{A})\) be a measurable space. We denote by \(\mathcal{M}\) the set of all finite signed measures on \((X, \mathcal{A})\).

a) If \(a, b \in \mathbb{R}\) and \(\mu, \nu \in \mathcal{M}\), we define 
\[(a\mu + b\nu)(A) = a\mu(A) + b\nu(A)\]
for \(A \in \mathcal{A}\). Show that \(a\mu + b\nu \in \mathcal{M}\) and that \(\mathcal{M}\) is a vector space over \(\mathbb{R}\) with this linear structure.

b) For \(\mu \in \mathcal{M}\) define \(\|\mu\| = |\mu|(X)\). Show that \(\mu \mapsto \|\mu\|\) defines a norm on \(\mathcal{M}\).

c) Show that the vector space \(\mathcal{M}\) equipped with the norm defined in (b) is a Banach space.

Problem 2: Let \(\nu\) be a complex measure on a measurable space \((X, \mathcal{A})\).

a) Let \(\nu_r = \nu^+ - \nu^-\) and \(\nu_i = \nu^+_i - \nu^-_i\) be the Jordan decompositions of the real part \(\nu_r\) and the imaginary part \(\nu_i\) of \(\nu\). Show that if \(|\nu|\) denotes the total variation of \(\nu\), then \(\nu^+_r, \nu^-_r, \nu^+_i, \nu^-_i \leq |\nu|\).

b) We say that a measurable function \(f\) on \((X, \mathcal{A})\) is \(\nu\)-integrable if it is integrable with respect to \(|\nu|\). So if we denote the space of these functions \(f\) by \(L^1(\nu)\), then \(L^1(\nu) = L^1(|\nu|)\). Show that if \(f \in L^1(\nu)\), then \(f\) is integrable with respect to each of the measures \(\nu^+_r, \nu^-_r, \nu^+_i, \nu^-_i\) and so
\[
\int f \, d\nu := \int f \, d\nu^+_r - \int f \, d\nu^-_r + i \int f \, d\nu^+_i - i \int f \, d\nu^-_i
\]
is well-defined.

c) Suppose \(\mu\) is a \(\sigma\)-finite positive measure on \((X, \mathcal{A})\) such that \(\nu << \mu\) and let \(g = d\nu/d\mu\) be the Radon-Nikodym derivative of \(\nu\) with respect to \(\mu\). Show that if \(f \in L^1(\nu)\), then \(fg \in L^1(\mu)\) and
\[
\int f \, d\nu = \int fg \, d\mu.
\]

Problem 3:

a) Let \(f : \mathbb{R}^n \to \mathbb{C}\) and \(g : \mathbb{R}^n \to \mathbb{C}\) be Borel measurable functions on \(\mathbb{R}^n\). Show that the function \(F : \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}\) defined as
\[
F(x, y) = f(x - y)g(y)
\]
for \(x, y \in \mathbb{R}^n\)
is also Borel measurable.
b) Let \( f: \mathbb{R}^n \to \mathbb{C} \) and \( g: \mathbb{R}^n \to \mathbb{C} \) be (Lebesgue) integrable functions. Show that then the convolution of \( f \) and \( g \) given by

\[
(f \ast g)(x) := \int_{\mathbb{R}^n} f(x - y)g(y) \, d\lambda_n(y)
\]

is well-defined for almost every \( x \in \mathbb{R}^n \) and that

\[
\|f \ast g\|_1 \leq \|f\|_1 \cdot \|g\|_1.
\]

c) Show that if \( f: \mathbb{R}^n \to \mathbb{C} \) and \( g: \mathbb{R}^n \to \mathbb{C} \) are integrable functions, then

\[
(f \ast g)(x) = (g \ast f)(x)
\]

for almost every \( x \in \mathbb{R}^n \).

**Problem 4:** Let \( X \) be a topological space, and \( f: X \to X \) be a continuous map. A Borel measure \( \mu \) on \( X \) is called \( f \)-invariant if \( f_* \mu = \mu \), or equivalently, if

\[
\mu(f^{-1}(B)) = \mu(B)
\]

for all Borel sets \( B \subseteq X \).

Consider two \( f \)-invariant Borel probability measures \( \nu \) and \( \mu \) on \( X \) and let \( \nu = \nu_s + \nu_a \) be the Lebesgue decomposition of \( \nu \) with respect to \( \mu \), where \( \nu_s \perp \mu \) and \( \nu_a \ll \mu \). Show that then \( \nu_s \) and \( \nu_a \) are also \( f \)-invariant.