Here $\nu^+ \perp \nu^-$. 

By assumption $\nu^+ \perp \nu^-$ and so $|\nu| = \nu^+ + \nu^-$. We can apply Case II to the pairs $\nu^+$ and $\nu^-$. 

This gives Lebesgue decomposition:

\[ \nu^+ = \nu^{+} + \nu^{+} \quad \text{and} \quad \nu^- = \nu^{-} + \nu^{-} \quad \text{where} \quad \nu^{+}, \nu^{-} \perp \mu. \]

\[ d\nu^+ = f^+ d\mu, \quad d\nu^- = f^- d\mu, \]

\[ f^+: \mathbb{R} \to [0, \infty) \text{ almost everywhere}. \]

Since $\nu^+ \perp \nu^-$, we have $\nu^+ \perp \nu^-$ and $\nu^- \perp \nu^-$. 

Hence $\nu^+ = \nu^+ + \nu^-$ and $\nu^- = \nu^- + \nu^-$. 

\[ \nu = \nu^+ + \nu^- \quad \text{and} \quad \nu^{+} = \nu^{+} + \nu^{+} \quad \text{and} \quad \nu^{-} = \nu^{-} + \nu^{-} \quad \text{are well-defined signed measures and} \]

\[ \nu^+ \perp \nu^- \quad \text{(exercise!) and} \quad \nu^- \perp \nu^+. \]

\[ \mu \perp \nu^+ \quad \text{and} \quad \mu \perp \nu^- \quad \text{and} \quad \mu \perp \nu. \]

Proof of the existence of Lebesgue decomposition is complete.

**Uniqueness (Outline)**

**Case I.** $\nu$ finite signed measure.

$\mu$ finite signed measure.

Suppose we have two Lebesgue decompositions of $\nu$ with $\mu$.

\[ \nu = \nu^+ + \nu^- \quad \text{and} \quad \nu = \nu^{+} + \nu^{-} \]

where $\nu^{+}, \nu^{-} \perp \mu$ and $d\nu^+ = f^+ d\mu$, $d\nu^- = f^- d\mu$.

Then $\nu^+, \nu^-, \nu^{+}, \nu^{-}$ are also finite, and

\[ \nu^+ = \nu^- \quad \text{and} \quad \nu^+ = \nu^-. \]

So, $\nu^+ = \nu^-$. Hence $\nu = 0$ which implies...
\[ J_1 = J_2, \quad S_1 = S, \quad r_2 = r, \quad \mu - a.s. \]

**Case II (general case):**

\[ H = \text{finite signed measure,} \]
\[ \mu = \text{finite, non-negative measure.} \]

Partition \( X \) into countably many measurable sets on which \( \nu \) and \( \mu \) are finite and apply considerations of Case I on each set of partition.

---

**Complex measures**

**\((X, \mathcal{A})\) measurable space.** A complex measure \( \nu \) on \((X, \mathcal{A})\) is a function

\[ \nu : \mathcal{A} \to \mathbb{C} \text{ s.t.} \]

1. \( \nu(\emptyset) = 0 \).
2. \( \nu \) is countably additive, i.e., if \( A_n \in \mathcal{A} \), \( n \in \mathbb{N} \), are pairwise disjoint, then

\[ \nu \left( \bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} \nu(A_n). \]

(Convergence of series is part of the statement.)

If \( \nu \) is a complex measure, then

\[ \nu = \nu_+ + i\nu_- \quad \text{where} \quad \nu_+ \quad \text{(real part of} \ \nu) \quad \text{and} \quad \nu_- \quad \text{(imaginary part of} \ \nu) \quad \text{are signed measures, given by} \]

\[ \nu_+(A) := \text{Re} \ \nu(A) \quad \text{for} \ A \in \mathcal{A}. \]
\[ \nu_-(A) := \text{Im} \ \nu(A) \]

Since \( \nu \) only takes finite values (i.e., values in \( \mathbb{C} \)), \( \nu_+ \) and \( \nu_- \) take values in \( \mathbb{R} \) and are hence finite measures.
(5) \( \nu \) = \( \nu^+ - \nu^- \) \quad \text{Lebesgue decoup.}

\[ \nu^+ (X), \nu^- (X) < \infty \quad \text{so} \]

\[ |\nu^+ (X) - \nu^- (X)| < \infty \]

Similarly, \( |\nu^- (X)| < \infty \)

\text{Radon-Nikodym Thm.} \quad \text{for} \quad \text{coupled} \quad \mu \quad \text{and} \quad \nu.

\((X, \mathcal{A})\) \quad \text{meas. space}, \quad \nu < \infty \quad \text{finite Pos.}

\text{meas., } \nu \quad \text{coupled} \quad \text{meas. on} \quad (X, \mathcal{A}).

Then \( \nu = f + g \), where

\[ f \perp \mathcal{A} \quad (\text{i.e., } f \perp \mathcal{A}, f \perp 1_X) \quad \text{and} \]

\[ g < \infty \quad (\text{i.e., } g \perp \mathcal{A}, g \perp 1_X) \]

Move over! There ex. \( f \in \mathbb{L}(\mu) \) s.t.

\[ d\nu = d\mu \]

The meas. \( f, g \) are unique, and \( f \) is unique \( \mu \)-a.e.

\textbf{Proof:} \text{Apply Radon-Nikodym Thm. to the finite signed meas. } \nu \quad \text{and} \quad \nu^+.

Let \( \nu \) be a complex meas. on \((X, \mathcal{A})\).

Then we define the total variation \( |\nu| \)

of \( \nu \) as the set function \( |\nu| : \mathcal{A} \to [0, \infty] \)
given by

\[ |\nu| (A) = \sup \left\{ \sum_{n=1}^{\infty} |\nu(A_n)| : A_n \text{ point. disj., } \bigcup_{n \in \mathbb{N}} A_n = A \right\} \]

\textbf{Notes:}

For a signed meas. \( \nu \) this agrees with the total variation \( |\nu| = |\nu^+| + |\nu^-| \)
defined by using Lebesgue decoup. \( \nu = \nu^+ - \nu^- \).
Prop. 16. If \( V \) is a countably measurable on
\((X, \mathcal{U})\), then \( |V| \) is a finite pos. meas.
on \((X, \mathcal{U})\).

Proof: We can find a finite pos. meas. \( \mu \) on
\((X, \mathcal{U}) \) s.t. \( V = \mu \) (take \( \mu = 1_{V} + 1_{V'} \),
for example).

Then by Radon-Nikodym theorem, \( \exists f \in L^{1}(\mu) \)
s.t. \( dV = f \, d\mu \).

Claim
\[ |V|(A) = \int_{A} |f| \, d\mu \quad \text{for } A \in \mathcal{U}, \]
(i.e. \( d|V| = |f| \, d\mu \) and so \(|V|\) is a finite
pos. meas.).

Let \( A \in \mathcal{U} \) be open and \( A = \bigcup A_{n} \)
be a nees. partition of \( A \).

Then
\[ \sum_{n=1}^{\infty} |V(A_{n})| = \sum_{n=1}^{\infty} \left| \int_{A_{n}} f \, d\mu \right| \leq \sum_{n=1}^{\infty} \int_{A_{n}} |f| \, d\mu . \]

Hence
\[ |V|(A) = \int_{A} |f| \, d\mu , \]
so \( |V|(A) \leq \int_{A} |f| \, d\mu \).

Now: Let \( A \in \mathcal{U} \) and \( \epsilon > 0 \) be open.
Since simple functions are dense in \( L^{1}(\mu) \),
there exist
\[ s = \sum_{k=1}^{K} c_{k} \chi_{B_{k}} \quad \text{with } c_{1}, \ldots, c_{K} \in \mathcal{C},
B_{1}, \ldots, B_{K} \in \mathcal{U}, \text{ and } \]
s.t. \( \|f-s\|_{1} < \epsilon \).
Let \( B = B_1 \cup \ldots \cup B_k \),
\[ A_0 = A \cap B^c, \quad A_i = A \cap B_i, \quad \text{for } i = 1, \ldots, k. \]

Thus,
\[ \sum_{i=0}^{k} |\nu(A_i)| = \sum_{i=0}^{k} \int \left| \frac{1}{s} - 1 \right| \, ds d\mu_1 \]
\[ \geq \sum_{i=0}^{k} \int_{A_i} |s_1 - s_2| \, ds d\mu_1 \]
\[ \geq \sum_{i=0}^{k} \int_{A_i} |s_1 - s_2| \, ds d\mu_1 - 2\varepsilon = \int_{A} |s_1| \, d\mu_1 - 2\varepsilon. \]

Since \( \varepsilon > 0 \) or b. we conclude
\[ \nu(A) \geq \int_{A} |s_1| \, d\mu_1. \]

\[ \text{Review of basic functional analysis over } F = \mathbb{R} \text{ or } \mathbb{C} \]

\( X, Y \) normed vector spaces, \( T : X \to Y \)
linear w.r.t.
Then \( T \) is called **bounded** if there exists \( C > 0 \) s.t.
\[ \|T(x)\| \leq C \|x\| \text{ for all } x \in X. \]

Thm. \( X, Y \) normed vector spaces, \( T : X \to Y \)
linear. TFAE:
i) \( T \) is continuous,
ii) \( T \) is continuous at 0 \( \in \mathbb{F} \),
iii) \( T \) is bounded.
(b) (Proof: i) \(\Rightarrow\) (iii): Obvious.

ii) \(\Rightarrow\) iii): Note that \(T(0) = 0\).

So if \(T\) is cont. at 0, then there ex.
\(\delta > 0\) s.t.
\[\|x\| \leq \delta \quad \rightarrow \quad \|T(x)\| \leq 1 \quad \text{for}\ x \in X.\]

If \(u \in X, u \neq 0\), is orl. def. 
\[x = \frac{\delta}{\|u\|} u. \quad \text{Then} \quad \|x\| = \delta; \quad \text{so} \quad \|T(u)\| \leq 1.\]

\[\|T(u)\| \leq \delta \|u\|.
\]

So \(T\) is bdd.

iii) \(\Rightarrow\) i): Let \(T\) is bdd. Then there ex. \(C > 0\) s.t.
\[\|T(x)\| \leq C\|x\| \quad \text{for all} \quad x \in X.\]

Then
\[\|T(x) - T(x_0)\| = \|T(x - x_0)\| \leq C\|x - x_0\|.
\]

So \(T\) is Lipschitz and hence continuous.

If \(T: X \rightarrow \mathbb{R}\) is a bounded linear op.
("bounded operator") we define
\[\|T\| : = \sup \left\{ \frac{\|T(x)\|}{\|x\|} : x \in X \setminus \{0\} \right\}\]

\[= \inf \left\{ C > 0 : \|T(x)\| \leq C\|x\| \quad \text{for all} \quad x \in X \right\}\]

\[= \sup \left\{ \|T(x)\| : x \in X, \|x\| \leq 1 \right\}.\]

\(\|T\|\) is the operator norm of \(T\).
9. \( L(\mathcal{X}, \mathcal{Y}) = \{ T : \mathcal{X} \to \mathcal{Y} \mid T \text{ is bounded linear operator} \} \)

\( L(\mathcal{X}, \mathcal{Y}) \) equipped with operator norm

is a normed vector space over \( \mathbb{F} \)

\((\alpha S + \beta T)(x) = \alpha S(x) + \beta T(x), \ x \in \mathcal{X} \)

\[ \| T \| = 0 \iff T = 0 \]

\[ \| \alpha T \| = |\alpha| \cdot \| T \| \quad \alpha \in \mathbb{F} \]

\[ \| S + T \| \leq \| S \| + \| T \| \]

The operator norm is submultiplicative:

if \( T \in L(\mathcal{X}, \mathcal{Y}), S \in L(\mathcal{Y}, \mathcal{Z}) \),

then \( S \circ T \in L(\mathcal{X}, \mathcal{Z}) \), and

\[ \| S \circ T \| \leq \| S \| \cdot \| T \| \]

Prop.: Let \( \mathcal{X} \) and \( \mathcal{Y} \) be normed spaces over \( \mathbb{F} \). If \( \mathcal{Y} \) is complete, then

\( L(\mathcal{X}, \mathcal{Y}) \) is complete

(equiv.: if \( \mathcal{X} \) is a Banach space, then

\( L(\mathcal{X}, \mathcal{Y}) \) is a Banach space).

Proof (outline): Let \( \{ T_n \} \) be a Cauchy seq. in \( L(\mathcal{X}, \mathcal{Y}) \). Then for each \( x \in \mathcal{X} \), \( \{ T_n(x) \} \) is a Cauchy seq. in \( \mathcal{Y} \) (given that \( \mathcal{Y} \) is complete).

Define \( T(x) = \lim_{n \to \infty} T_n(x) \).

Then one can show \( T \in L(\mathcal{X}, \mathcal{Y}) \) and \( T_n \to T \). \( \Box \)
Bounded linear functionals and dual spaces

A bounded linear map \( f : X \to F \) is called a bounded linear functional.

\( f \in \mathcal{L}(X,F) \).

The space \( \mathcal{L}(X,F) \) is called the space of bounded linear functionals.

\( X^* = \{ f : X \to F : \text{f is a bounded linear functional} \} \)

\[ \|f\| = \sup \{ |f(x)| : \|x\| \leq 1 \} \]

is a norm on \( X^* \) (operator norm).

If \( X \) is a Banach space, then \( X^* \) is also a Banach space (by Prop. 2 above).

Thm. (Hahn–Banach Thm; simple version)

Let \( X \) be a normed vector space, \( M \subseteq X \) a (linear) subspace, and \( f : M \to F \) a bounded linear functional. Then there exists a bounded linear functional \( F : X \to F \) such that \( F|_M = f \) and \( \|F\| = \|f\| \).

Proof: Fellouh, p. 157–158
Cor. For each \( x \in X \) we have
\[
\|x\| = \sup \left\{ |f(x)| : f \in X^*, \|f\| \leq 1 \right\}
\]
and this sup. is attained as a max.

Proof: \( x \neq 0 \) and \( f(x) \neq 0 \) for \( f \in X^* \) with \( \|f\| = 1 \). Let \( M = \{ f(x) : f \in X^*, \|f\| \leq 1 \} \) and define \( g : \mathbb{R} \rightarrow \mathbb{F} \) by \( g(2x) = 2 \|x\|, \forall \mathbb{R} \). Then \( g(2x) = 1 \|x\| \), \( 2 \|x\| = \|2x\| \),

\[\|x\| = \sup \left\{ |f(x)| : f \in X^*, \|f\| \leq 1 \right\}\]

Note: \( \|f\| = \sup \left\{ |f(x)| : x \in X, \|x\| \leq 1 \right\} \)

by def. of \( \|f\| \). In general, sup. is not attained as max.

The dual of \( L^p \)
\[L^p(X, \mathcal{A}, \mu)\]

is the space \( \{ f : f \in L^p(X, \mathcal{A}, \mu) : \int |f|^p \, d\mu < \infty \} \)

where \( f = 0 \text{ a.e.} \) means \( \int |f|^p \, d\mu = 0 \)

\[\|f\|_p = \left( \int |f|^p \, d\mu \right)^{1/p} \]

for \( f \in L^p \), \( p = 1, 2, \ldots, \infty \)

for \( f \) in \( L^p \), \( f = 0 \text{ a.e.} \) means \( \int |f|^p \, d\mu = 0 \).
22. Let \( q \) be the conjugate exp. of \( p \):
\[
\frac{1}{p} + \frac{1}{q} = 1 \quad \text{equiv.} \quad q = \frac{p}{p-1} \quad (p \neq 1, p = \infty).
\]
If \( g \in L^q(\mu) \), we define a linear function \( \Phi_g \) by
\[
\Phi_g : L^p(\mu) \to \mathbb{C} \quad \text{by setting}
\]
\[
\Phi_g(f) = \int f g \, d\mu.
\]

\( \Phi_g \) is bounded linear function, because \( |\Phi_g(f)| = \left| \int f g \, d\mu \right| \leq \|f\|_p \|g\|_q \quad \text{and} \quad \|\Phi_g\| \leq \|g\|_q. \)

\[\text{Thm. (L-L^q duality)}\]
Let \( 1 \leq p < \infty \) and \( q \in (1, \infty] \) be the conjugate exp. of \( (X, \mathcal{A}, \mu) \) a \( \sigma \)-finite measure space.
Then for each bounded linear funct. \( \Phi : L^p(\mu) \to \mathbb{C} \) there exists \( g \in L^q(\mu) \) s.t. \( \Phi = \Phi_g \).
Moreover, \( g \) is unique \( \mu \)-a.e. (i.e., \( \Phi = \Phi_g = \Phi_{\tilde{g}} \)), \( \mu \)-a.e.) and \( \|\Phi\| = \|g\|_q. \)
23. \textbf{Rem.} \(1\) If we define a map
\[
T : L^q(\mu) \to L^p(\mu) + \mathbb{P} \text{ by }
T(g) := \Phi g \quad \text{for } g \in L^q(\mu),
\]
then
\begin{enumerate}[(i)]
  \item \(T\) is well-defined. (\(\Phi g\) only depends on \([g]\))
  \item \(T\) is linear.
  \item \(T\) is surjective by Thm.
  \item \(\|T(g)\| = \|\Phi g\| = \|g\|_q\), i.e. \(T\) is norm-preserving and hence injective (\(\|T(g)\|^p = 0 \implies \|g\|^q = 0 \implies [g] = 0\)).
\end{enumerate}

So \(T\) is an isomorphism between
\[L^q(\mu) \text{ and } L^p(\mu) + \mathbb{P},\]
and this isomorphism is
\[L^p(\mu) + = L^q(\mu) \quad \text{for } 1 \leq p \leq \infty.\]

\section*{2. In general: \(L^\infty(\mu) \neq L^1(\mu)\)}
\textit{(examples in discussion session.)}

\textbf{Proof of Thm. (Outline)} \(1 \leq p \leq \infty, \quad 1 \leq q \leq \infty.
\underline{Uniqueness:} Suppose \(\Phi g = \Phi = \Phi g\) for \(g, \hat{g} \in L^q = L^q(\mu)\).

Then
\[
\int f g \, d\mu = \int f \hat{g} \, d\mu \quad \text{for all } f \in L^p
\]
equiv.
\[
\int f (g - \hat{g}) \, d\mu = 0 \quad \text{for all } f \in L^p
\]
\[\implies \int 1_E (g - \hat{g}) \, d\mu = 0 \quad \text{whenever } E \in \mathcal{L}, \quad \mu(E) < \infty
\]
\textit{(exercise!)}
\[g - \hat{g} = 0 \mu\text{-a.e. equiv. } g = \hat{g} \mu\text{-a.e.} \]
24. **Existence:** \( \Phi \in (L^p)^* \) for \( p \) and \( \lambda > 0 \).

**Case I:** \( \mu \) is a finite measure, i.e., \( \mu(X) < \infty \).

For \( A \in \mathcal{A} \), we have
\[
\int X_A d\mu = \int X_A d\mu = \mu(A) < \infty,
\]
so \( X_A \in L^p \).

Define
\[
\nu(A) = \Phi(X_A) \in \mathbb{C}.
\]
Then \( \nu \) is a complex measure on \( (X, \mathcal{A}) \):

1. \( \nu(\emptyset) = \Phi(X_\emptyset) = \Phi(0) = 0 \)
2. \( \nu \) is countably additive:
   Let \( A_n \in \mathcal{A} \) be pairwise disjoint and \( A = \bigcup_n A_n \).
   Define \( B_n = A \setminus U A_n \).
   Then \( X_{B_n} \to X_A \) in \( L^p \); indeed:
   \[
   \|X_{B_n} - X_A\|_p = \int |X_{B_n} - X_A| d\mu
   = \int |X_{A \setminus B_n}| d\mu = \int X_{A \setminus B_n} d\mu = \mu(A \setminus B_n)
   \to 0 \text{ as } n \to \infty, \text{ because } A \setminus B_n \to \emptyset.
\]

By continuity of \( \Phi \), we have
\[
\nu(A) = \Phi(X_A) = \lim_{n \to \infty} \Phi(X_{B_n})
= \lim_{n \to \infty} \Phi(X_A + \cdots + X_{A_n})
= \lim_{n \to \infty} \nu(A_1) + \cdots + \nu(A_n).
\]
\[ \Phi (h) = \lim_{n \to \infty} \Phi (s_n) = \lim_{n \to \infty} \int s_n g \, dp = \lim_{n \to \infty} \int h g \, dp \]

We have \( \gamma < \mu \) indeed, i.e. \( A \in \mathcal{K} \) and \( \mu (A) = 0 \), i.e., \( \chi_A = 0 \) \( \mu \)-a.e., i.e.,

\[ \Phi (\chi_A) = \Phi (0) = \Phi ([0]) = 0. \]

So by Radon-Nikodym there ex. \( g \in L^1 \) s.t.

\[ \Phi (\chi_A) = \gamma (A) = \int_A g \, dp \]

By linearity this extends to all simple functions, i.e.,

\[ \Phi (s) = \int s g \, dp \quad \text{for all simple functions} \quad s. \]

Let \( h \in L^\infty \subset L^p \). Since simple functions are dense in \( L^\infty \), there ex. a \( s_n \) of simple functions s.t.

\[ s_n \to h \quad \text{in } L^\infty \quad \text{equiv. } \| s_n - h \|_{L^\infty} \to 0 \]

Hence

\[ \| s_n - h \|_p = \left( \int |s_n - h|^p \, dp \right)^{1/p} = \left( \int \| s_n - h \|_{L^\infty}^p \, dp \right)^{1/p} \]

\[ = \mu (\mathbb{R}) \| s_n - h \|_{L^\infty} \to 0. \]

So \( s_n \to h \) in \( L^p \) and so

\[ \Phi (h) = \lim_{n \to \infty} \Phi (s_n) = \lim_{n \to \infty} \int s_n g \, dp \]

\[ = \lim_{n \to \infty} \int h g \, dp \]

\[ = \int h g \, dp \]
Note:
\[
\limsup_{n \to \infty} \left| \int f_n g \, d\mu - \int f g \, d\mu \right| \\
\limsup_{n \to \infty} \int |f_n - f| \, |g| \, d\mu \\
\limsup_{n \to \infty} \|f_n - f\|_2 \int |g| \, d\mu = 0.
\]
So
\[
(3) \quad \overline{\Phi}(h) = \int f g \, d\mu \quad \text{for all } h \in L^2.
\]
Claim \( g \in L^q \).

**Subcase 1:** \( 1 < p < \infty \). Then \( 1 < q < \infty \).

There \( \exists \) a non-negative function \( \alpha \) on \( \mathbb{X} \) s.t. \( |\alpha| = 1 \) and \( q = \alpha \|g\|_q \) (exercise!)

Let \( E_n = \{ |g| \leq n \} \in \mathcal{A} \) and
\[
f_n := \alpha |g| : \mathbb{X} \to \mathbb{C} \in L^\infty \subset L^p
\]
So by (3):
\[
\int_{E_n} |g| \, d\mu = \left| \int_{E_n} f g \, d\mu \right| = |\overline{\Phi}(f_n)|
\]
\[
\leq \|\overline{\Phi}\| \|f_n\|_p \|g\|_{p(q-1)P}^{1/p} \|P\| = \frac{q}{q-1}
\]
\[
= \|\overline{\Phi}\| \left( \int_{E_n} |g| \, d\mu \right)^{1/q} \left( \int_{E_n} |g|^{q-1} \, d\mu \right)^{1/p}^{1/q}
\]
\[
\to \left( \int_{E_n} |g| \, d\mu \right)^{1-1/p} \leq \|\overline{\Phi}\|.
\]

\[
(\int_{E_n} |g| \, d\mu)^{1/q} \leq \|\overline{\Phi}\|.
\]
(27) So by MCT \((X_{E_n} \rightarrow 1 = X_{\mathbb{F}})\)
\[
\|g\|_q = \lim_{n \to \infty} \left( \int_{E_n} |g|^q \right)^{1/q} \leq \|\mathbf{g}\|_q < \infty.
\]
So indeed, \(g \in L^q\) and \(\|g\|_q \leq \|\mathbf{g}\|_q\).

In particular, \(f \in L^p \rightarrow \Phi_f(f) = \int f \, g \, d\mu\)
defines a bounded linear functional on \(L^p\).

By (2)
\[
\Phi(s) = \Phi_g(s) \quad \text{for all simple functions}
\]
functions one dense in \(L^p\), we have \(\mathbf{g} = \Phi g\).

Moreover
\[
\|g\|_q \leq \|\mathbf{g}\| \leq \|g\|_q \quad \text{so,} \quad \|\mathbf{g}\| = \|g\|_q.
\]

Subcase 2: \(p = 1, \ q = \infty\).

Based on (3) above, show
\(\|g\|_1 \leq \|\mathbf{g}\|\) and completes the proof.

As in Subcase 2.

The existence proof in Case 1 is complete.

Case 2: (general case) \(\mu\) is \(\infty\)-finite.

One uses the usual idea:

Then exists, sets \(E_n \rightarrow \mathbb{F}\) s.t.
\[
\mu(E_n) < \infty \quad \text{for} \quad n \in N.
\]

One considers the restriction
\[
\Phi_n = \Phi|L^p(\mu, E_n) = \int f \, L^p(\mu) \quad f = 0 \quad \text{on } E_n \quad \mu\text{-a.e.}
\]
and uses Case 1 (details in homework!) \(\square\).
25. **Notation:** \( B = B(a, r) = \{ x \in X : d(a, x) < r \} \)

an open ball centered at \( a \in X \) with radius \( r > 0 \)
in a metric space \( (X, d) \).

\( a > 0 \), \( \forall B \subseteq B(a, a +) \).

**Thm. (Basic 5B-covering lemma):** Let \( (X, d) \) be a metric space and \( B \) be a collection of balls in \( X \) with uniformly bounded radius. Then there exists a collection \( \tilde{B} \subseteq \tilde{B} \) of disjoint balls (a disjointed family) such that \( \bigcup \tilde{B} = \bigcup \tilde{B} \). Let \( B = B(a, r) \) and \( B = B(a, r') \)

are balls in \( X \) with \( B \cap B' = \emptyset \) and \( r' < \frac{1}{2} r \). Then \( B \subseteq 5 B' \).

Indeed, \( \forall x \in B \cap B' \) and if \( x \in B \), then

\[ d(a, x) < d(a, x) + d(x, a) + d(a, x) \]

\[ < r + 2r = 5r' \] hence \( x \in 5 B' \) and \( B \subseteq 5 B' \).

We now consider the collection

\[ M = \{ A \subseteq X : A \text{ disjointed family and } B \subseteq B \text{ meets a ball in } A \} \]

Then it holds one in \( A \) whose radius is at least half the radius of \( B \).
M is partially ordered by inclusion: $A_1 \subseteq A_2$.

Note that:

i) $M \neq \emptyset$. Indeed we can find a ball $B \in \mathcal{B}$ s.t.

radius of $B = \frac{1}{2} \sup \text{ radius of } B$ (a) 

Then $\bigcup \{B^i \in M\}$

ii) every chain $C$ in $M$ (i.e., every totally ordered subset of $M$) has an upper bound $\mathcal{A}$:

Take $\mathcal{A} = \bigcup \{A \in M : A \leq C\}$

Indeed, then

1) $\mathcal{A}$ consists of disjoint balls:

If $B_1, B_2 \in \mathcal{A}$, then $B_1 \neq B_2$. Hence ex.

If $A_1, A_2 \in C$ s.t. $B_1 \subseteq A_1 \cup A_2$.

Since $C$ is a chain $A_1 \leq A_2$ or $A_2 \leq A_1$, w.l.o.g.

$A_1 \leq A_2$. Then $B_1 \cup B_2 \subseteq C$ and so $B_1 \cap B_2 = \emptyset$.

2) if $B \in \mathcal{B}_2$ and $B$ meets a ball $B' \in \mathcal{A}$, then $B \subseteq A$, where $A \in C$.

Then $B$ meets a ball $B' \in \mathcal{A}$ with radius $B' = \frac{1}{2}$ radius $B$.

So $\mathcal{A} \in C$ and $\mathcal{A}$ is an upper bound for $C$ ($\mathcal{A} \in \mathcal{A}$ s.t. $\mathcal{A} \in C$).
By Zorn's lemma, \( M \) has a maximal element \( \mathcal{B} \in C \).

Then \( \mathcal{B} \):

i) is a disjoint family,

ii) every ball \( B \in \mathcal{B} \) meets a ball in \( \mathcal{B} \).

Otherwise, we can find \( B', B \in \mathcal{B} \) s.t.

\[ B \cap B' = \emptyset \]

for all \( B \in \mathcal{B} \) and radius \( D' = \frac{1}{2} \sup \text{ radius of } B \text{ balls disjoint from all balls in } \mathcal{B} \).

Then \( \mathcal{B}' := \mathcal{B} \cup \{ B' \} \) disjoint.

Let's take \( \mathcal{B}' \neq \mathcal{B} \).

1) \( B \) meets a ball in \( \mathcal{B}' \), then \( B \) meets a ball in \( \mathcal{B} \) and hence \( \mathcal{B} \subseteq \mathcal{B}' \), with radius \( \geq \frac{1}{2} \text{ radius of } B \).

2) \( B \) doesn't meet a ball in \( \mathcal{B}' \), but meets \( B' \). Then

radius \( B' \geq \frac{1}{2} \text{ radius of } B \) by definition of \( B' \).

So \( \mathcal{B}' \subseteq \mathcal{C} \) which contradicts maximality of \( \mathcal{B} \).

So \( \mathcal{B} \) is not a basis. Then by (iii)

\( B \) meets a ball in \( \mathcal{B} \). Since \( \mathcal{B} \subseteq \mathcal{C} \),

\( B \) meets a ball \( \hat{B} \in \mathcal{B} \) s.t.

radius \( \hat{B} \geq \frac{1}{2} \text{ radius of } B \).