Homework 8 (due: Fr., Dec. 1)

Problem 1: Let \((X, \mathcal{A}, \mu)\) and \((Y, \mathcal{B}, \nu)\) be \(\sigma\)-finite measures spaces. Suppose \(\alpha\) and \(\beta\) are two measures on the measurable space \((X \times Y, \mathcal{A} \otimes \mathcal{B})\) such that
\[
\alpha(A \times B) = \mu(A) \cdot \nu(B) = \beta(A \times B),
\]
whenever \(A \in \mathcal{A}, B \in \mathcal{B}\). Show that then \(\alpha = \beta\).

Problem 2: The purpose of this problem is to give an alternative approach to the construction of product measures. In the following, \((X, \mathcal{A}, \mu)\) and \((Y, \mathcal{B}, \nu)\) are two measures spaces.

a) Let \(C\) be the family of all sets \(M \subseteq X \times Y\) that can be written as a disjoint finite union of measurable rectangles, i.e.,
\[
M = (A_1 \times B_1) \cup \cdots \cup (A_n \times B_n),
\]
where \(n \in \mathbb{N}, A_1, \ldots, A_n \in \mathcal{A}, B_1, \ldots, B_n \in \mathcal{B}\), and \((A_i \times B_i) \cap (A_j \times B_j) = \emptyset\) for \(i \neq j\).

Show that \(C\) is an algebra on \(X \times Y\). Hint: Use a lemma discussed in class (Folland, p. 23, Prop. 1.7).

b) If \(M \in C\) is as in (1), define
\[
\omega(M) = \sum_{i=1}^{n} \mu(A_i) \nu(B_i).
\]
Show that \(\omega\) is well-defined and that \(\omega\) is a premeasure on the algebra \(C\). Hint: Represent \(\omega(M)\) as a double integral over \(\chi_M\). Justify the measurability of all relevant functions!

c) Show that there exists a measure \(\alpha\) on \(\mathcal{A} \otimes \mathcal{B}\) (the \(\sigma\)-algebra generated by the family of all measurable rectangles) such that
\[
\alpha(A \times B) = \mu(A) \cdot \nu(B),
\]
whenever \(A \in \mathcal{A}\) and \(B \in \mathcal{B}\).

Problem 3: For \(n \in \mathbb{N}\) we denote the \(\sigma\)-algebra of all Lebesgue measurable subsets of \(\mathbb{R}^n\) by \(\mathcal{L}_n\) and Lebesgue measure on \(\mathbb{R}^n\) by \(\lambda_n\).

Show that if \(n, k \in \mathbb{N}\) are arbitrary, then \(\mathcal{L}_n \otimes \mathcal{L}_k \subseteq \mathcal{L}_{n+k}\) and that the measure space \((\mathbb{R}^{n+k}, \mathcal{L}_{n+k}, \lambda_{n+k})\) is the completion of \((\mathbb{R}^{n+k}, \mathcal{L}_n \otimes \mathcal{L}_k, \lambda_n \times \lambda_k)\). Here we make the usual identification \(\mathbb{R}^{n+k} \cong \mathbb{R}^n \times \mathbb{R}^k\).

Problem 4: Let \(f: \mathbb{R}^2 \to \mathbb{R}\) be a real-valued function such that \(f_x = f(x, \cdot)\) is Borel measurable for each \(x \in \mathbb{R}\) and \(f^y = f(\cdot, y)\) is continuous for each \(y \in \mathbb{R}\). Show that then \(f\) is Borel measurable. Hint: Use a piecewise linear approximation and pass to the limit.