Homework 8 (due: Fr., Dec. 1)

Problem 1: Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measures spaces. Suppose α and β are two measures on the measurable space $(X \times Y, \mathcal{A} \otimes \mathcal{B})$ such that

$$\alpha(A \times B) = \mu(A) \cdot \nu(B) = \beta(A \times B),$$

whenever $A \in \mathcal{A}$, $B \in \mathcal{B}$. Show that then $\alpha = \beta$.

Problem 2: The purpose of this problem is to give an alternative approach to the construction of product measures. In the following, $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are two measures spaces.

a) Let $\mathcal{C}$ be the family of all sets $M \subseteq X \times Y$ that can be written as a disjoint finite union of measurable rectangles, i.e.,

$$M = (A_1 \times B_1) \cup \cdots \cup (A_n \times B_n), \tag{1}$$

where $n \in \mathbb{N}$, $A_1, \ldots, A_n \in \mathcal{A}$, $B_1, \ldots, B_n \in \mathcal{B}$, and $(A_i \times B_i) \cap (A_j \times B_j) = \emptyset$ for $i \neq j$.

Show that $\mathcal{C}$ is an algebra on $X \times Y$. Hint: Use a lemma discussed in class (Folland, p. 23, Prop. 1.7).

b) If $M \in \mathcal{C}$ is as in (1), define

$$\omega(M) = \sum_{i=1}^{n} \mu(A_i)\nu(B_i).$$

Show that ω is well-defined and that ω is a premeasure on the algebra $\mathcal{C}$. Hint: Represent $\omega(M)$ as a double integral over χ_M. Justify the measurability of all relevant functions!

c) Show that there exists a measure α on $\mathcal{A} \otimes \mathcal{B}$ (the σ-algebra generated by the family of all measurable rectangles) such that

$$\alpha(A \times B) = \mu(A) \cdot \nu(B),$$

whenever $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Problem 3: For $n \in \mathbb{N}$ we denote the σ-algebra of all Lebesgue measurable subsets of $\mathbb{R}^n$ by $\mathcal{L}_n$ and Lebesgue measure on $\mathbb{R}^n$ by λ_n.

Show that if $n, k \in \mathbb{N}$ are arbitrary, then $\mathcal{L}_n \otimes \mathcal{L}_k \subseteq \mathcal{L}_{n+k}$ and that the measure space $(\mathbb{R}^{n+k}, \mathcal{L}_{n+k}, \lambda_{n+k})$ is the completion of $(\mathbb{R}^{n+k}, \mathcal{L}_n \otimes \mathcal{L}_k, \lambda_n \times \lambda_k)$. Here we make the usual identification $\mathbb{R}^{n+k} \cong \mathbb{R}^n \times \mathbb{R}^k$.

Problem 4: Let $f \colon \mathbb{R}^2 \to \mathbb{R}$ be a real-valued function such that $f_x = f(x, \cdot)$ is Borel measurable for each $x \in \mathbb{R}$ and $f^y = f(\cdot, y)$ is continuous for each $y \in \mathbb{R}$. Show that then f is Borel measurable. Hint: Use a piecewise linear approximation and pass to the limit.