Homework 7 (due: Mo, Nov. 20)

Problem 1: Let $(X, \mathcal{A}, \mu)$ be a measure space.

a) If $f\colon X \to \mathbb{C}$ is measurable, recall that

$$\|f\|_\infty := \inf\{\lambda \in [0,\infty] : \mu(\{x \in X : |f(x)| > \lambda\}) = 0\}.$$

Show that this infimum is attained as a minimum and that $|f(x)| \leq \|f\|_\infty$ for μ-almost every $x \in X$.

b) Show that $\|\cdot\|_\infty$ is a norm on $L^\infty(\mu)$.

c) Show that L^∞ is complete: if $\{f_n\}$ is a Cauchy sequence in $L^\infty(\mu)$, then there exists a function $f \in L^\infty(\mu)$ such that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$.

Problem 2: (cf. Folland, p. 63, Prob. 34) Let $(X, \mathcal{A}, \mu)$ be a measure space, and let g, f, and f_n for $n \in \mathbb{N}$ be measurable functions on X with $g \in L^1(\mu)$ and $|f_n| \leq g$ for all $n \in \mathbb{N}$.

Show that if $f_n \to f$ in measure, then $f_n \to f$ in L^1.

Problem 3: (cf. Folland, p. 63, Prob. 36) Let $(X, \mathcal{A}, \mu)$ be a measure pace, $A_n \in \mathcal{A}$ with $\mu(A_n) < \infty$ for $n \in \mathbb{N}$, and $f \in L^1(\mu)$.

Show that if $\chi_{A_n} \to f$ in L^1, then there exists a set $A \in \mathcal{A}$ with $\mu(A) < \infty$ such that $f = \chi_A$ μ-a.e. on X.

Problem 4: (cf. Folland, p. 59, Prob. 21) Let $(X, \mathcal{A}, \mu)$ be a measure pace, $f \in L^1(\mu)$, and $f_n \in L^1(\mu)$ for $n \in \mathbb{N}$. Suppose that $f_n(x) \to f(x)$ as $n \to \infty$ for μ-a.e. $x \in X$. Show that then

$$f_n \to f \text{ in } L^1 \text{ if and only if } \int |f_n|\, d\mu \to \int |f|\, d\mu \text{ as } n \to \infty.$$