Homework 6 (due: Mo, Nov. 13)

Problem 1: Let $(X, \mathcal{A})$ be a measurable space, and $\mathcal{F}$ be a family of functions $f\colon X \to \mathbb{R}$. Suppose that $\mathcal{F}$ has the following properties:

(i) There exists a π-system $\mathcal{P} \subseteq \mathcal{A}$ with $X \in \mathcal{P}$ and $\sigma(\mathcal{P}) = \mathcal{A}$ such that $\chi_A \in \mathcal{F}$ for each $A \in \mathcal{P}$.

(ii) The family $\mathcal{F}$ is closed under linear combinations: if $f, g \in \mathcal{F}$ and $\alpha \in \mathbb{R}$, then $\alpha f + g \in \mathcal{F}$.

(iii) The family $\mathcal{F}$ is closed under monotone limits: if $f_n \in \mathcal{F}$ for $n \in \mathbb{N}$ and $f_n \nearrow f$, then $f \in \mathcal{F}$.

Show that then $\mathcal{F}$ contains *all* measurable functions $f\colon X \to \mathbb{R}$ (this statement is an example of a so-called *montone class theorem*). Hint: First show that $\chi_A \in \mathcal{F}$ for each $A \in \mathcal{A}$.

Problem 2: a) Let (X, d) be a metric space, $A \subseteq X$ compact, $B \subseteq X$ closed, and $A \cap B = \emptyset$. Show that then

$$\operatorname{dist}(A, B) := \inf\{d(a, b) : a \in A,\, b \in B\} > 0.$$

b) Let X be a locally compact Hausdorff space, $K \subseteq X$ compact, $U \subseteq X$ open, and $K \subseteq U$. Show that then there exists an open set $V \subseteq X$ with $K \subseteq V \subseteq \overline{V} \subseteq U$ such that $\overline{V}$ is compact.

Problem 3: Let $(X, \mathcal{A}, \mu)$ be a measure space.

a) Show that if $1 \le p < r < q \le \infty$, then $L^p(\mu) \cap L^q(\mu) \subseteq L^r(\mu)$. Hint: For the relevant estimate split $|f|^r$ into two terms.

b) Show that if $\mu(X) < \infty$, then $L^q(\mu) \subseteq L^p(\mu)$ whenever $1 \le p < q \le \infty$.

c) Give an example (for each pair $p < q$) showing that the inclusion $L^q(\mu) \subseteq L^p(\mu)$ is not true in general without the assumption $\mu(X) < \infty$.

Problem 4: Let $1 \le p < \infty$ and denote by $L^p(\mathbb{R})$ the L^p-space with Lebesgue measure on $\mathbb{R}$ as the underlying measure. Show that $L^p(\mathbb{R})$ is *separable* for $1 \le p < \infty$, i.e., $L^p(\mathbb{R})$ contains a countable dense subset. Hint: Consider suitable piecewise linear functions.