Homework 5 (due: Fr, Nov. 3)

Problem 1: Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces. Then the *product σ-algebra* $\mathcal{A} \otimes \mathcal{B}$ is defined to be the smallest σ-algebra on $X \times Y$ that contains all sets $A \times B$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

We denote the Borel σ-algebra on $\mathbb{R}^l$ by $\mathcal{B}_l$. Show that then $\mathcal{B}_n \otimes \mathcal{B}_k = \mathcal{B}_{n+k}$ for all $n, k \in \mathbb{N}$.

Problem 2: (cf. Folland, Prop. 2.11) Let $(X, \mathcal{A}, \mu)$ be a complete measure space.

a) Suppose $f\colon X \to \overline{\mathbb{R}}$ is measurable and $g\colon X \to \overline{\mathbb{R}}$ is a function such that $f = g$ μ-almost everywhere. Show that then g is also measurable.

b) Suppose $f_n\colon X \to \overline{\mathbb{R}}$ is measurable for $n \in \mathbb{N}$ and that for some function $f\colon X \to \overline{\mathbb{R}}$ we have $f = \lim\limits_{n\to\infty} f_n$ μ-almost everywhere, i.e., there exists a set $N \in \mathcal{A}$ with $\mu(N) = 0$ such that $\lim\limits_{n\to\infty} f_n(x)$ exists and $f(x) = \lim\limits_{n\to\infty} f_n(x)$ for each $x \in X \setminus N$. Show that then f is measurable.

Problem 3: A function $f\colon \mathbb{R} \to (-\infty, +\infty]$ is called *lower semicontinuous* if the set $\{x \in \mathbb{R} : f(x) > t\}$ is open for each $t \in \mathbb{R}$.

a) Show that a function $f\colon \mathbb{R} \to (-\infty, +\infty]$ is lower semicontinuous if and only if $\liminf\limits_{n\to\infty} f(x_n) \geq f(a)$, whenever $a \in \mathbb{R}$ and $\{x_n\}$ is a sequence in $\mathbb{R}$ with $x_n \to a$.

b) Show that every lower semicontinuous function $f\colon \mathbb{R} \to (-\infty, +\infty]$ is Borel measurable.

c) Show that if $I = [a, b] \subseteq \mathbb{R}$ is a finite closed interval, then every lower semicontinuous function $f\colon \mathbb{R} \to (-\infty, +\infty]$ attains a minimum on I.

Problem 4: Show that a function $f\colon \mathbb{R} \to (-\infty, +\infty]$ is lower semicontinuous if and only if there exists a sequence $\{g_n\}$ of continuous real-valued functions on $\mathbb{R}$ such that $g_n \nearrow f$.

Hint: In order to define g_n use quantities such as $f(a) + n|x - a|$.