Homework 4 (due: Fr, Oct. 27)

Problem 1: The purpose of this problem is to characterize translation-invariant measures on $\mathbb{R}^n$. We know that Lebesgue measure λ on $\mathbb{R}^n$ is such a measure. Suppose μ is another measure on $\mathbb{R}^n$ defined on the σ-algebra $\mathcal{L}$ of all (Lebesgue) measurable subsets of $\mathbb{R}^n$. We assume that $c_0 := \mu((0,1]^n) < \infty$ and that $\mu(M) = \mu(t+M)$, whenever $t \in \mathbb{R}^n$ and $M \in \mathcal{L}$.

a) Show that $\mu(Q) = c_0\lambda(Q)$ for all *h-cubes* Q of sidelength $1/k$ with $k \in \mathbb{N}$, i.e., for all sets of the form $Q = t + (0, 1/k]^n$, where $t \in \mathbb{R}$.

b) Show that $\mu(R) = c_0\lambda(R)$ for all h-rectangles $R \subseteq \mathbb{R}^n$ whose sidelengths are rational numbers.

c) Show that $\mu(R) = c_0\lambda(R)$ for all h-rectangles $R \subseteq \mathbb{R}^n$.

d) Show that $\mu(B) = c_0\lambda(B)$ for all Borel sets $B \subseteq \mathbb{R}^n$.

e) Show that $\mu(M) = c_0\lambda(M)$ for all $M \in \mathcal{L}$.

Problem 2: Suppose $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a C^1-diffeomorphism, i.e., a bijection that is C^1-smooth and has a C^1-smooth inverse. Show that $M \subseteq \mathbb{R}^n$ is Lebesgue measurable if and only if $f(M)$ is Lebesgue measurable.

Problem 3: Let λ be Lebesgue measure on $\mathbb{R}^n$ and $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation.

a) Show that if we set $\mu_T(M) = \lambda(T(M))$, then μ_T is defined for each Lebesgue measurable set $M \subseteq \mathbb{R}^n$ and gives a translation-invariant measure on $\mathbb{R}^n$. Show that this implies that there exists a unique number $\Delta_T \geq 0$ such that $\mu_T(M) = \Delta_T\lambda(M)$ for each Lebesgue measurable $M \subseteq \mathbb{R}^n$.

b) Let $S\colon \mathbb{R}^n \to \mathbb{R}^n$ be another linear transformation. Show that then

$$\Delta_{S\circ T} = \Delta_S \cdot \Delta_T.$$

c) Show that $\Delta_T = 0$ whenever $\det(T) = 0$.

Problem 4: a) Let $e_1 = (1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1)$ be the standard basis vectors in $\mathbb{R}^n$. Then there exists a unique linear transformation $L\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $L(e_1) = e_1 + e_2$ and $L(e_i) = e_i$ for $i = 2, \dots, n$ (why?). Let $Q_0 = (0, 1]^n$.

Show that there exist disjoint Borel sets $A, B \subseteq L(Q_0)$ with $L(Q_0) = A \cup B$ such that for some $t \in \mathbb{R}^n$ we have $A \cup (t + B) = Q_0$ and $A \cap (t + B) = \emptyset$. Use this to show that $\Delta_L = 1$ for the number defined in Problem 3(a).

b) Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation. Show that T can be written as a finite composition of the following types of linear transformations S:

Type I: S permutes the standard basis vectors;

Type II: $S(e_i) = \alpha_i e_i$ with $\alpha_i \neq 0$ for $i = 1, \ldots, n$;

Type III: $S = L$, where L is as in (a).

Hint: First show this for linear transformations corresponding to elementary row operations on matrices.

c) Show that $\Delta_T = |\det(T)|$ for each linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$, or equivalently, $\lambda(T(M)) = |\det(T)|\lambda(M)$ for each Lebesgue measurable set $M \subseteq \mathbb{R}^n$.