Homework 2 (due: Fr, Oct. 13)

Problem 1: Let $X$ be a set and $\mathcal{F}$ be a family of subsets of $X$. Then $\mathcal{F}$ is called a $\pi$-system on $X$ if it is closed under finite intersections: if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$. The family $\mathcal{F}$ is called a $\lambda$-system on $X$ if it has the following properties:

(i) $X \in \mathcal{F}$,
(ii) if $A, B \in \mathcal{F}$ and $B \subseteq A$, then $A \setminus B \in \mathcal{F}$,
(iii) if $A_n \in \mathcal{F}$ for $n \in \mathbb{N}$ and $A_n \nearrow$, then $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{F}$.

a) Show that if $\mathcal{S}$ is any family of subsets of $X$, then there exists a smallest $\lambda$-system on $X$, denoted $\lambda(\mathcal{S})$, that contains $\mathcal{S}$.

b) Suppose $\mathcal{S}$ is a $\pi$-system on $X$ and $\mathcal{F} = \lambda(\mathcal{S})$ is the smallest $\lambda$-system containing $\mathcal{S}$. Show that $\mathcal{F} = \lambda(\mathcal{S})$ is a $\pi$-system. Hint: First show that for fixed $S \in \mathcal{S}$ the family of all $A \subseteq X$ with $A \cap S \in \lambda(\mathcal{S})$ is a $\lambda$-system.

c) Prove Dynkin’s $\pi$-$\lambda$-Theorem: if $\mathcal{S}$ is a $\pi$-system and $\mathcal{F}$ a $\lambda$-system on $X$ that contains $\mathcal{S}$, then $\sigma(\mathcal{S}) \subseteq \mathcal{F}$, i.e., $\mathcal{F}$ contains the $\sigma$-algebra generated by $\mathcal{S}$.

Problem 2: Let $\mu$ and $\nu$ be measures on a measure space $(X, \mathcal{F})$ and $\mathcal{S} \subseteq \mathcal{F}$ a $\pi$-system that generates $\mathcal{F}$ (i.e., $\mathcal{F} = \sigma(\mathcal{S})$). Suppose that $\mu(\mathcal{S}) = \nu(\mathcal{S})$ for all $S \in \mathcal{S}$.

a) Show that if $\mu$ and $\nu$ are finite measures with $\mu(X) = \nu(X)$, then $\mu = \nu$. Give an example that shows that this implication is not true without the requirement that $\mathcal{S}$ is a $\pi$-system.

b) Prove the following extension of (a): Suppose there exist sets $S_n \in \mathcal{S}$ with $\mu(S_n) = \nu(S_n) < \infty$ for $n \in \mathbb{N}$ and $\bigcup_{n \in \mathbb{N}} S_n = X$. Then $\mu = \nu$.

Problem 3: Let $\mu$ be a Borel measure on $\mathbb{R}^n$, i.e., a measure defined on the Borel $\sigma$-algebra $\mathcal{B}$ on $\mathbb{R}^n$. Suppose that $\mu(K) < \infty$ for each compact set $K \subseteq \mathbb{R}^n$.

a) Show that if $F \subseteq \mathbb{R}^n$ is a Borel set with $\mu(F) < \infty$ and we define
$$\nu(A) = \mu(F \cap A)$$
for $A \in \mathcal{B}$, then $\nu$ is a finite Borel measure on $\mathbb{R}^n$.

b) Show that $\mu$ is regular in the following sense:
$$\mu(A) = \inf\{\mu(U) : U \subseteq \mathbb{R}^n \text{ open and } A \subseteq U\}$$
for each Borel set $A \subseteq \mathbb{R}^n$ (outer regularity) and
$$\mu(A) = \sup\{\mu(K) : K \subseteq \mathbb{R}^n \text{ compact and } K \subseteq A\}$$
for each Borel set $A \subseteq \mathbb{R}^n$ (inner regularity).
Problem 4: Let $(X, \mathcal{A})$ be a measure space, and $\mu$ and $\nu$ be measures on $(X, \mathcal{A})$. The measure $\nu$ is called absolutely continuous with respect to $\mu$ if every $\mu$-null set is also a $\nu$-null set (i.e., $A \in \mathcal{A}$ and $\mu(A) = 0$ imply $\nu(A) = 0$).

Show that if $\nu$ is a finite measure, then $\nu$ is absolutely continuous with respect to $\mu$ if and only if the following condition is true: for each $\epsilon > 0$ there exists $\delta > 0$ such that for all $A \in \mathcal{A}$ we have the implication $\mu(A) < \delta \Rightarrow \nu(A) < \epsilon$.

Show that this equivalence is not true without the assumption that $\nu$ is finite.