Homework 2 (due: Fr, Oct. 13)

Problem 1: Let X be a set and $\mathcal{F}$ be a family of subets of X. Then $\mathcal{F}$ is called a π-*system* on X if it is closed under finite intersections: if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$. The family $\mathcal{F}$ is called a λ-*system* on X if it has the following properties:

(i) $X \in \mathcal{F}$,

(ii) if $A, B \in \mathcal{F}$ and $B \subseteq A$, then $A \setminus B \in \mathcal{F}$,

(iii) if $A_n \in \mathcal{F}$ for $n \in \mathbb{N}$ and $A_n \nearrow$, then $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{F}$.

a) Show that if $\mathcal{S}$ is any family of subsets of X, then there exists a smallest λ-system on X, denoted $\lambda(\mathcal{S})$, that contains $\mathcal{S}$.

b) Suppose $\mathcal{S}$ is a π-system on X and $\mathcal{F} = \lambda(\mathcal{S})$ is the smallest λ-system containing $\mathcal{S}$. Show that $\mathcal{F} = \lambda(\mathcal{S})$ is a π-system. Hint: First show that for fixed $S \in \mathcal{S}$ the family of all $A \subseteq X$ with $A \cap S \in \lambda(\mathcal{S})$ is a λ-system.

c) Prove *Dynkin's* π-λ-*Theorem*: if $\mathcal{S}$ is a π-system and $\mathcal{F}$ a λ-system on X that contains $\mathcal{S}$, then $\sigma(\mathcal{S}) \subseteq \mathcal{F}$, i.e., $\mathcal{F}$ contains the σ-algebra generated by $\mathcal{S}$.

Problem 2: Let μ and ν be measures on a measure space $(X, \mathcal{F})$ and $\mathcal{S} \subseteq \mathcal{F}$ a π-system that generates $\mathcal{F}$ (i.e., $\mathcal{F} = \sigma(\mathcal{S})$). Suppose that $\mu(S) = \nu(S)$ for all $S \in \mathcal{S}$.

a) Show that if μ and ν are finite measures with $\mu(X) = \nu(X)$, then $\mu = \nu$. Give an example that shows that this implication is not true without the requirement that $\mathcal{S}$ is a π-system.

b) Prove the following extension of (a): Suppose there exist sets $S_n \in \mathcal{S}$ with $\mu(S_n) = \nu(S_n) < \infty$ for $n \in \mathbb{N}$ and $\bigcup_{n \in \mathbb{N}} S_n = X$. Then $\mu = \nu$.

Problem 3: Let μ be a Borel measure on $\mathbb{R}^n$, i.e., a measure defined on the Borel σ-algebra $\mathcal{B}$ on $\mathbb{R}^n$. Suppose that $\mu(K) < \infty$ for each compact set $K \subseteq \mathbb{R}^n$.

a) Show that if $F \subseteq \mathbb{R}^n$ is a Borel set with $\mu(F) < \infty$ and we define

$$\nu(A) = \mu(F \cap A) \text{ for } A \in \mathcal{B},$$

then ν is a finite Borel measure on $\mathbb{R}^n$.

b) Show that μ is *regular* in the following sense:

$$\mu(A) = \inf\{\mu(U) : U \subseteq \mathbb{R}^n \text{ open and } A \subseteq U\}$$

for each Borel set $A \subseteq \mathbb{R}^n$ (*outer regularity*) and

$$\mu(A) = \sup\{\mu(K) : K \subseteq \mathbb{R}^n \text{ compact and } K \subseteq A\}$$

for each Borel set $A \subseteq \mathbb{R}^n$ (*inner regularity*).

Problem 4: Let $(X, \mathcal{A})$ be a measure space, and μ and ν be measures on $(X, \mathcal{A})$. The measure ν is called *absolutely continuous with respect to* μ if every μ-null set is also a ν-null set (i.e., $A \in \mathcal{A}$ and $\mu(A) = 0$ imply $\nu(A) = 0$).

Show that if ν is a finite measure, then ν is absolutely continuous with respect to μ if and only if the following condition is true: for each $\epsilon > 0$ there exists $\delta > 0$ such that for all $A \in \mathcal{A}$ we have the implication $\mu(A) < \delta \Rightarrow \nu(A) < \epsilon$.

Show that this equivalence is not true without the assumption that ν is finite.