Homework 1 (due: Fr, Oct. 6)

Note: Only typed-up homework will be excepted, preferably prepared with a system adapted to mathematical typesetting such as Latex. Homework is due every Friday before class starts (I also accept it if I find it in my office when I return from class on Fridays). Late homework is not accepted. I am happy to give an extension under exceptional circumstances if I am asked 24 hours before the deadline.

Problem 1: Let $\mathcal{A}$ be a σ-algebra on a set X. Suppose that $\mathcal{A}$ has infinitely many elements.

a) Show that then there exists an infinite sequence of pairwise disjoint non-empty sets in $\mathcal{A}$.

b) Show that $\mathcal{A}$ is uncountable.

Problem 2: Let μ be a finite Borel measure on $\mathbb{R}^n$, i.e., a measure with $\mu(\mathbb{R}^n) < \infty$ defined on the Borel subsets of $\mathbb{R}^n$.

Consider the family $\mathcal{A}$ of all Borel subsets M of $\mathbb{R}^n$ such that for each $\epsilon > 0$ there exists a compact set K and and an open set U with $K \subseteq M \subseteq U$ and $\mu(U \setminus K) < \epsilon$.

a) Show that $\mathcal{A}$ is an algebra.

b) Show that $\mathcal{A}$ is closed under increasing limits: if $A_n \in \mathcal{A}$ for $n \in \mathbb{N}$ and $A_n \nearrow$, then $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$. Conclude that $\mathcal{A}$ is a σ-algebra.

c) Show that each Borel set $M \subseteq \mathbb{R}^n$ belongs to $\mathcal{A}$.

Problem 3: Let $(X, \mathcal{A}, \mu)$ be a measure space, and $A_n \in \mathcal{A}$ for $n \in \mathbb{N}$.

a) Consider the set

$$A = \{x \in X : x \in A_n \text{ for infinitely many } n \in \mathbb{N}\}$$

Show that $A \in \mathcal{A}$.

b) Suppose in addition that $\displaystyle\sum_{n=1}^{\infty} \mu(A_n) < \infty$. Show that then $\mu(A) = 0$.

Problem 4: Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is non-decreasing, i.e, for $x, y \in \mathbb{R}$ we have the implication

$$x \le y \implies f(x) \le f(y).$$

Show that if $A \subset \mathbb{R}$ is a Borel set, then so is $f(A) = \{f(x) : x \in A\}$.

Hint: Consider the family $\mathcal{A}$ consisting of all Borel sets $A \subseteq \mathbb{R}$ such that $f(A)$ is a Borel set.