Remarks for the solutions of Homework 3

I will present a detailed solution for Problem 3 and give some hints for the other problems.

**Problem 1.** Part (a) essentially reduces to the distributive law for numbers in $[0, \infty]$. In part (b) one decomposes the intervals appearing in the $i$th coordinate of the rectangles by using all points that appear as endpoints of any of the intervals in the $i$th coordinate. Part (d) uses a covering argument based on the fact that an open cover of a compact set has a finite subcover. It will be explained in the TA session.

**Problem 2.** One proves the implication chain (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (i). Most of the implications are straightforward based on covering arguments with careful book keeping of volume (=Lebesgue measure). For example, to prove (ii) $\Rightarrow$ (iii) one has to find a cover of a rectangle $R$ by cubes whose total volume is only slightly larger than the volume of $R$. For this one increases $R$ slightly to an $h$-rectangle $R'$ that is the product of intervals with rational side lengths. Then one considers the least common denominator $N$ of the rational numbers representing these side lengths. Then $R'$ can be covered by pairwise disjoint translates of the $h$-cube $(0, 1/N]^n$. The total volume of these cubes is equal to the volume of $R'$ and hence very close to the volume of $R$.

**Problem 3.** We will use the following notation. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are two points in $\mathbb{R}^n$, written as row vectors with coordinates $x_i \in \mathbb{R}$ and $y_i \in \mathbb{R}$, respectively, where $i = 1, \ldots, n$, then we denote by

$$|x - y| := ((x_1 - y_1)^2 + \cdots + (x_n - y_n)^2)^{1/2}$$

their Euclidean distance. If $a \in \mathbb{R}^n$ and $r > 0$ we denote by

$$B(a, r) := \{x \in \mathbb{R}^n : |x - a| < r\}$$

the open ball and by

$$\overline{B}(a, r) := \{x \in \mathbb{R}^n : |x - a| \leq r\}$$

the closed ball of radius $r$ centered at $a$.

In order to prove the statement in Problem 3, we will establish several claims.

**Claim 1.** Let $A \subseteq \mathbb{R}^n$ be a bounded set and $g : \mathbb{R}^n \to \mathbb{R}$ be a $C^1$-smooth function. Then $g$ is Lipschitz on $A$, i.e., there exists a constant $L \geq 0$ such that

$$|g(x) - g(y)| \leq L|x - y|$$

for all $x, y \in A$. 

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First note that because \( A \) is bounded, there exists a number \( R > 0 \) such that 
\( A \subseteq Q := [-R, R]^n \). The cube \( Q \) is a \textit{convex set}. This mean that if \( x, y \in Q \) and 
\( t \in [0, 1] \), then \( tx + (1-t)y \in Q \). To see this, note that if \( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in Q \), then we have \( x_i, y_i \in [-R, R] \) for \( i = 1, \ldots, n \). If \( t \in [0, 1] \) and \( i \in \{1, \ldots, n\} \), then the \( i \)th coordinate of the point \( tx + (1-t)y \) is equal to 
\( tx_i + (1-t)y_i \). Since \( x_i, y_i \in [-R, R] \) and \( t \in [0, 1] \), we have \( tx_i + (1-t)y_i \in [-R, R] \) 
for each \( i = 1, \ldots, n \). Hence \( tx + (1-t)y \in Q \).

We denote by \( \partial_i g \) the partial derivative of \( g \) with respect to the \( i \)th coordinate in \( \mathbb{R}^n \), where \( i \in \{1, \ldots, n\} \). Since \( g \) is \( C^1 \)-smooth, \( \partial_i g \) is a continuous function on \( \mathbb{R}^n \). Now the set \( Q = [-R, R]^n \) is closed and bounded, and hence compact. So \( \partial_i g \) attains its maximum and minimum on \( Q \). In particular, \( \partial_i g \) is bounded on \( Q \). This means that for each \( i \in \{1, \ldots, n\} \) there exists a constant \( M_i \geq 0 \) such that 
\[ |\partial_i g(u)| \leq M_i \]
for all \( u \in Q \).

Define \( L := (M_1^2 + \cdots + M_n^2)^{1/2} \) and let \( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in A \subseteq Q \) be arbitrary. We consider the function \( h: [0, 1] \to \mathbb{R} \) defined as 
\[ h(t) = g(tx + (1-t)y) \]
for \( t \in [0, 1] \). By the chain rule, this is a differentiable function with derivative 
\[ h'(t) = (x_1 - y_1)\partial_1 g(u_t) + \cdots + (x_n - y_n)\partial_n g(u_t). \]
for \( t \in [0, 1] \), where \( u_t := tx + (1-t)y \). This formula shows that \( h' \) is a continuous function on \( [0, 1] \). So by the fundamental theorem of calculus we have 
\[ h(1) - h(0) = \int_0^1 h'(t) \, dt. \]

Note that \( u_t = tx + (1-t)y \) in \( Q \) for \( t \in [0, 1] \), because \( Q \) is convex. Hence by (1) we have 
\[ |\partial_i g(u_t)| \leq M_i \]
for all \( i \in \{1, \ldots, n\} \) and \( t \in [0, 1] \).

Now (2), (3), (4), and the Cauchy-Schwarz inequality imply that 
\[ |g(x) - g(y)| = |h(1) - h(0)| = \left| \int_0^1 h'(t) \, dt \right| \leq \max_{t \in [0,1]} |h'(t)| \leq M_1|x_1 - y_1| + \cdots + M_n|x_n - y_n| \leq (M_1^2 + \cdots + M_n^2)^{1/2} \cdot ((x_1 - y_1)^2 + \cdots + (x_n - y_n)^2)^{1/2} = L|x - y|. \]

This is the desired inequality. Claim 1 follows.
Claim 2. Let $A \subseteq \mathbb{R}^n$ be a bounded set and $f : \mathbb{R}^n \to \mathbb{R}^n$ be a $C^1$-smooth map. Then $f$ is Lipschitz on $A$, i.e., there exists a constant $L \geq 0$ such that

$$|f(x) - f(y)| \leq L|x - y|$$

for all $x, y \in A$.

The difference to Claim 1 is that the map in question is not real-valued, but $\mathbb{R}^n$-valued. Claim 2 can easily be derived from Claim 1. Indeed, let $f_1, \ldots, f_n : \mathbb{R}^n \to \mathbb{R}$ be the component functions of $f$, i.e., for $x \in \mathbb{R}^n$ the vector $f(x) \in \mathbb{R}^n$ has the form

$$f(x) = (f_1(x), \ldots, f_n(x)).$$

Since $f$ is $C^1$-smooth, each function $f_i$ for $i \in \{1, \ldots, n\}$ is $C^1$-smooth on $\mathbb{R}^n$. In particular, $f_i$ is Lipschitz on our given bounded set $A \subseteq \mathbb{R}^n$ by Claim 1. Hence for each $i \in \{1, \ldots, n\}$ there exists $L_i \geq 0$ such that

$$|f_i(x) - f_i(y)| \leq L_i|x - y|$$

for all $x, y \in \mathbb{R}^n$.

Now define $L := (L_1^2 + \cdots + L_n^2)^{1/2}$. Then for all $x, y \in A$ we have

$$|f(x) - f(y)| = |(f_1(x) - f_1(y), \ldots, f_n(x) - f_n(y))|$$

$$= (|f_1(x) - f_1(y)|^2 + \cdots + |f_n(x) - f_n(y)|^2)^{1/2}$$

$$\leq (L_1^2|x - y|^2 + \cdots + L_n^2|x - y|^2)^{1/2}$$

$$= L|x - y|,$$

as desired.

Recall that a null-set $N \subseteq \mathbb{R}^n$ is a measurable set in $\mathbb{R}^n$ that has Lebesgue measure 0. We will use the characterization of null-sets that was given in (iv) of Problem 2.

Claim 3. Let $N \subseteq \mathbb{R}^n$ be a null-set, and $f : N \to \mathbb{R}^n$ be a Lipschitz map. Then $f(N)$ is also a null set.

Under the assumptions as in the claim, there exists a constant $L > 0$ such that

$$|f(x) - f(y)| \leq L|x - y|$$

for all $x, y \in N$. Note that we can assume that the Lipschitz constant $L$ is positive here, because we can always increase the Lipschitz constant if necessary.

In order to show that $f(N)$ is a null-set, we want to verify condition (iv) in Problem 2. So let $\epsilon > 0$ be arbitrary. Since $N$ is a null-set, there exist balls $B_k := B(a_k, r_k)$ with $a_k \in \mathbb{R}^n$ and $r_k > 0$ for $k \in \mathbb{N}$ such that

$$N \subseteq \bigcup_{k \in \mathbb{N}} B_k$$

(5)
We may assume that $B_m \cap N \neq \emptyset$ for each $m \in \mathbb{N}$, because if $B_m \cap N = \emptyset$, then we can just delete the ball $B_m$ from the countable cover $\{B_k\}_{k \in \mathbb{N}}$ of $N$ without affecting (5).

So for each $k \in \mathbb{N}$ we can pick a point $b_k \in B_k \cap N$. Then if $x \in B_k \cap N$ is arbitrary, we have

$$|x - b_k| \leq |x - a_k| + |a_k - b_k| < 2r_k.$$ 

So if $c_k := f(b_k)$, then

$$|f(x) - c_k| = |f(x) - f(b_k)| \leq L|x - b_k| < 2Lr_k.$$ 

Since $x \in B_k \cap N$ was arbitrary, we conclude

$$f(N \cap B_k) \subseteq B(c_k, 2Lr_k).$$ 

So if we define $B'_k := B(c_k, s_k)$ with $s_k := 2Lr_k$ for $k \in \mathbb{N}$, then

$$f(N) = f\left(\bigcup_{k \in \mathbb{N}} (N \cap B_k)\right) = \bigcup_{k \in \mathbb{N}} f(N \cap B_k) \subseteq \bigcup_{k \in \mathbb{N}} B'_k,$$ 

and

$$\sum_{k=1}^{\infty} s_k^n = (2L)^n \sum_{k=1}^{\infty} r_k^n < \epsilon.$$ 

Since $\epsilon > 0$ was arbitrary, we conclude that $f(N)$ is a null-set, as desired.

**Claim 4.** Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a $C^1$-smooth map and $N \subseteq \mathbb{R}^n$ be a null-set. Then $f(N)$ is a null-set.

This is the statement of Problem 3. To prove it, we consider $N_k := N \cap \overline{B}(0, k)$ for $k \in \mathbb{N}$. Then $N_k \subseteq N$, and so $N_k$ is a null-set, because subsets of null-sets are null-sets. Moreover, $N_k \subseteq \overline{B}(0, k)$, and so $N_k$ is a bounded set. Claim 2 and Claim 3 now imply that $f(N_k)$ is a null-set.

Note that $N = \bigcup_{k \in \mathbb{N}} N_k$ and so

$$f(N) = f\left(\bigcup_{k \in \mathbb{N}} N_k\right) = \bigcup_{k \in \mathbb{N}} f(N_k).$$ 

Since each set $f(N_k)$, $k \in \mathbb{N}$, is a null-set and hence measurable, the set $f(N)$ is measurable as the countable union of measurable sets. Moreover, countable subadditivity of Lebesgue measure $\lambda$ on $\mathbb{R}^n$ shows that

$$0 \leq \lambda(f(N)) = \lambda\left(\bigcup_{k \in \mathbb{N}} f(N_k)\right) \leq \sum_{k=1}^{\infty} \lambda(f(N_k)) = \sum_{k=1}^{\infty} 0 = 0.$$
It follows that $\lambda(f(N)) = 0$. Hence $f(N)$ is a null-set, as desired.

Note that by an argument very similar to the one in the previous proof one can show that a countable union of null-sets is again a null-set.

**Problem 4.** First consider $H = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n = 0\}$. This is a null-set, because if $\epsilon > 0$ is arbitrary and $M_k := (2k)^{n-1}2^{k+1}$ for $k \in \mathbb{N}$, then

$$H \subseteq \bigcup_{k \in \mathbb{N}} [-k, k]^{n-1} \times [-\epsilon/M_k, \epsilon/M_k]$$

and so

$$\lambda(H) \leq \sum_{k=1}^{\infty} \lambda([-k, k]^{n-1} \times [-\epsilon/M_k, \epsilon/M_k]) = \sum_{k=1}^{\infty} (2k)^{n-1}2^{k+1} = \sum_{k=1}^{\infty} 2^{-k} \epsilon = \epsilon.$$

An arbitrary affine hyperplane $P$ is the image of $H_0$ under a suitable affine map $f : \mathbb{R}^n \to \mathbb{R}^n$. Such an affine map is $C^1$-smooth (actually $C^\infty$-smooth). Hence $P$ is a null-set by Problem 3. This implies that every subset $N$ of $P$ is also a null-set.