Remarks for the solutions of Homework 3

I will present a detailed solution for Problem 3 and give some hints for the other problems.

Problem 1. Part (a) essentially reduces to the distributive law for numbers in $[0, \infty]$. In part (b) one decomposes the intervals appearing in the ith coordinate of the rectangles by using all points that appear as endpoints of any of the intervals in the ith coordinate. Part (d) uses a covering argument based on the fact that an open cover of a compact set has a finite subcover. It will be explained in the TA session.

Problem 2. One proves the implication chain (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (i). Most of the implications are straightforward based on covering arguments with careful book keeping of volume (=Lebesgue measure). For example, to prove (ii) $\Rightarrow$ (iii) one has to find a cover of a rectangle R by cubes whose total volume is only slightly larger than the volume of R. For this one increases R slightly to an h-rectangle R' that is the product of intervals with rational side lengths. Then one considers the least common denominator N of the rational numbers representing these side lengths. Then R' can be covered by pairwise disjoint translates of the h-cube $(0, 1/N]^n$. The total volume of these cubes is equal to the volume of R' and hence very close to the volume of R.

Problem 3. We will use the following notation. If $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ are two points in $\mathbb{R}^n$, written as row vectors with coordinates $x_i \in \mathbb{R}$ and $y_i \in \mathbb{R}$, respectively, where $i = 1, \dots, n$, then we denote by

$$|x - y| := ((x_1 - y_1)^2 + \cdots + (x_n - y_n)^2)^{1/2}$$

their Euclidean distance. If $a \in \mathbb{R}^n$ and $r > 0$ we denote by

$$B(a, r) := \{x \in \mathbb{R}^n : |x - a| < r\}$$

the open ball and by

$$\overline{B}(a, r) := \{x \in \mathbb{R}^n : |x - a| \leq r\}$$

the closed ball of radius r centered at a.

In order to prove the statement in Problem 3, we will establish several claims.

Claim 1. Let $A \subseteq \mathbb{R}^n$ be a bounded set and $g \colon \mathbb{R}^n \to \mathbb{R}$ be a C^1-smooth function. Then g is Lipschitz on A, i.e., there exists a constant $L \geq 0$ such that

$$|g(x) - g(y)| \leq L|x - y|$$

for all $x, y \in A$.

First note that because A is bounded, there exists a number $R > 0$ such that $A \subseteq Q := [-R, R]^n$. The cube Q is a *convex set*. This mean that if $x, y \in Q$ and $t \in [0, 1]$, then $tx + (1-t)y \in Q$. To see this, note that if $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in Q$, then we have $x_i, y_i \in [-R, R]$ for $i = 1, \dots, n$. If $t \in [0, 1]$ and $i \in \{1, \dots, n\}$, then the ith coordinate of the point $tx + (1-t)y$ is equal to $tx_i + (1-t)y_i$. Since $x_i, y_i \in [-R, R]$ and $t \in [0, 1]$, we have $tx_i + (1-t)y_i \in [-R, R]$ for each $i = 1, \dots, n$. Hence $tx + (1-t)y \in Q$.

We denote by $\partial_i g$ the partial derivative of g with respect to the ith coordinate in $\mathbb{R}^n$, where $i \in \{1, \dots, n\}$. Since g is C^1-smooth, $\partial_i g$ is a continuous function on $\mathbb{R}^n$. Now the set $Q = [-R, R]^n$ is closed and bounded, and hence compact. So $\partial_i g$ attains its maximum and minimum on Q. In particular, $\partial_i g$ is bounded on Q. This means that for each $i \in \{1, \dots, n\}$ there exists a constant $M_i \geq 0$ such that

$$|\partial_i g(u)| \leq M_i \tag{1}$$

for all $u \in Q$.

Define $L := (M_1^2 + \cdots + M_n)^{1/2}$ and let $x = (x_1, \dots, x_n), y = (y_1, \dots y_n) \in A \subseteq Q$ be arbitrary. We consider the function $h\colon [0, 1] \to \mathbb{R}$ defined as

$$h(t) = g(tx + (1-t)y)$$

for $t \in [0, 1]$. By the chain rule, this is a differentiable function with derivative

$$h'(t) = (x_1 - y_1)\partial_1 g(u_t) + \cdots + (x_n - y_n)\partial_n g(u_t). \tag{2}$$

for $t \in [0, 1]$, where $u_t := tx + (1-t)y$. This formula shows that h' is a continuous function on $[0, 1]$. So by the fundamental theorem of calculus we have

$$h(1) - h(0) = \int_0^1 h'(t)\, dt. \tag{3}$$

Note that $u_t = tx + (1-t)y$ in Q for $t \in [0, 1]$, because Q is convex. Hence by (1) we have

$$|\partial_i g(u_t)| \leq M_i \tag{4}$$

for all $i \in \{1, \dots, n\}$ and $t \in [0, 1]$.

Now (2), (3), (4), and the Cauchy-Schwarz inequality imply that

$$\begin{aligned}
|g(x) - g(y)| = |h(1) - h(0)| &= \left| \int_0^1 h'(t)\, dt \right| \\
&\leq \max_{t \in [0,1]} |h'(t)| \\
&\leq M_1|x_1 - y_1| + \cdots + M_n|x_n - y_n| \\
&\leq (M_1^2 + \cdots + M_n^2)^{1/2} \cdot ((x_1 - y_1)^2 + \cdots + (x_n - y_n)^2)^{1/2} \\
&= L|x - y|.
\end{aligned}$$

This is the desired inequality. Claim 1 follows.

Claim 2. Let $A \subseteq \mathbb{R}^n$ be a bounded set and $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-smooth map. Then f is Lipschitz on A, i.e., there exists a constant $L \geq 0$ such that

$$|f(x) - f(y)| \leq L|x - y|$$

for all $x, y \in A$.

The difference to Claim 1 is that the map in question is not real-valued, but $\mathbb{R}^n$-valued. Claim 2 can easily be derived from Claim 1. Indeed, let $f_1, \dots, f_n\colon \mathbb{R}^n \to \mathbb{R}$ be the component functions of f, i.e., for $x \in \mathbb{R}^n$ the vector $f(x) \in \mathbb{R}^n$ has the form

$$f(x) = (f_1(x), \dots, f_n(x)).$$

Since f is C^1-smooth, each function f_i for $i \in \{1, \dots, n\}$ is C^1-smooth on $\mathbb{R}^n$. In particular, f_i is Lipschitz on our given bounded set $A \subseteq \mathbb{R}^n$ by Claim 1. Hence for each $i \in \{1, \dots, n\}$ there exists $L_i \geq 0$ such that

$$|f_i(x) - f_i(y)| \leq L_i|x - y|$$

for all $x, y \in \mathbb{R}^n$.

Now define $L := (L_1^2 + \cdots + L_n^2)^{1/2}$. Then for all $x, y \in A$ we have

$$\begin{aligned}
|f(x) - f(y)| &= |(f_1(x) - f_1(y), \dots, f_n(x) - f_n(y))| \\
&= \big((f_1(x) - f_1(y))^2 + \cdots + (f_n(x) - f_n(y))^2\big)^{1/2} \\
&\leq (L_1^2|x - y|^2 + \cdots + L_n^2|x - y|^2)^{1/2} \\
&= L|x - y|,
\end{aligned}$$

as desired.

Recall that a *null-set* $N \subseteq \mathbb{R}^n$ is a measurable set in $\mathbb{R}^n$ that has Lebesgue measure 0. We will use the characterization of null-sets that was given in (iv) of Problem 2.

Claim 3. Let $N \subseteq \mathbb{R}^n$ be a null-set, and $f\colon N \to \mathbb{R}^n$ be a Lipschitz map. Then $f(N)$ is also a null set.

Under the assumptions as in the claim, there exists a constant $L > 0$ such that

$$|f(x) - f(y)| \leq L|x - y|$$

for all $x, y \in N$. Note that we can assume that the Lipschitz constant L is positive here, because we can always increase the Lipschitz constant if necessary.

In order to show that $f(N)$ is a null-set, we want to verify condition (iv) in Problem 2. So let $\epsilon > 0$ be arbitrary. Since N is a null-set, there exist balls $B_k := B(a_k, r_k)$ with $a_k \in \mathbb{R}^n$ and $r_k > 0$ for $k \in \mathbb{N}$ such that

$$N \subseteq \bigcup_{k \in \mathbb{N}} B_k \tag{5}$$

and

$$\sum_{k=1}^{\infty} r_k^n < \frac{\epsilon}{2^n L^n}. \tag{6}$$

We may assume that $B_m \cap N \neq \emptyset$ for each $m \in \mathbb{N}$, because if $B_m \cap N = \emptyset$, then we can just delete the ball B_m from the countable cover $\{B_k\}_{k\in\mathbb{N}}$ of N without affecting (5).

So for each $k \in \mathbb{N}$ we can pick a point $b_k \in B_k \cap N$. Then if $x \in B_k \cap N$ is arbitrary, we have

$$|x - b_k| \leq |x - a_k| + |a_k - b_k| < 2r_k.$$

So if $c_k := f(b_k)$, then

$$|f(x) - c_k| = |f(x) - f(b_k)| \leq L|x - b_k| < 2Lr_k.$$

Since $x \in B_k \cap N$ was arbitrary, we conclude

$$f(N \cap B_k) \subseteq B(c_k, 2Lr_k).$$

So if we define $B'_k := B(c_k, s_k)$ with $s_k := 2Lr_k$ for $k \in \mathbb{N}$, then

$$f(N) = f\Bigg(\bigcup_{k\in\mathbb{N}} (N \cap B_k)\Bigg) = \bigcup_{k\in\mathbb{N}} f(N \cap B_k) \subseteq \bigcup_{k\in\mathbb{N}} B'_k,$$

and

$$\sum_{k=1}^{\infty} s_k^n = (2L)^n \sum_{k=1}^{\infty} r_k^n < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we conclude that $f(N)$ is a null-set, as desired.

Claim 4. Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-smooth map and $N \subseteq \mathbb{R}^n$ be a null-set. Then $f(N)$ is a null-set.

This is the statement of Problem 3. To prove it, we consider $N_k := N \cap \overline{B}(0,k)$ for $k \in \mathbb{N}$. Then $N_k \subseteq N$, and so N_k is a null-set, because subsets of null-sets are null-sets. Moreover, $N_k \subseteq \overline{B}(0,k)$, and so N_k is a bounded set. Claim 2 and Claim 3 now imply that $f(N_k)$ is a null-set.

Note that $N = \bigcup_{k\in\mathbb{N}} N_k$ and so

$$f(N) = f\Bigg(\bigcup_{k\in\mathbb{N}} N_k\Bigg) = \bigcup_{k\in\mathbb{N}} f(N_k).$$

Since each set $f(N_k)$, $k \in \mathbb{N}$, is a null-set and hence measurable, the set $f(N)$ is measurable as the countable union of measurable sets. Moreover, countable subadditivity of Lebesgue measure λ on $\mathbb{R}^n$ shows that

$$0 \leq \lambda(f(N)) = \lambda\Bigg(\bigcup_{k\in\mathbb{N}} f(N_k)\Bigg) \leq \sum_{k=1}^{\infty} \lambda(f(N_k)) = \sum_{k=1}^{\infty} 0 = 0.$$

It follows that $\lambda(f(N)) = 0$. Hence $f(N)$ is a null-set, as desired.

Note that by an argument very similar to the one in the previous proof one can show that a countable union of null-sets is again a null-set.

Problem 4. First consider $H = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n = 0\}$. This is a null-set, because if $\epsilon > 0$ is arbitrary and $M_k := (2k)^{n-1}2^{k+1}$ for $k \in \mathbb{N}$, then

$$H \subseteq \bigcup_{k \in \mathbb{N}} [-k, k]^{n-1} \times [-\epsilon/M_k, \epsilon/M_k]$$

and so

$$\begin{aligned}\lambda(H) &\leq \sum_{k=1}^{\infty} \lambda([-k, k]^{n-1} \times [-\epsilon/M_k, \epsilon/M_k]) \\ &= \sum_{k=1}^{\infty} (2k)^{n-1}(2\epsilon/M_k) = \sum_{k=1}^{\infty} \epsilon 2^{-k} = \epsilon.\end{aligned}$$

An arbitrary affine hyperplane P is the image of H_0 under a suitable affine map $f\colon \mathbb{R}^n \to \mathbb{R}^n$. Such an affine map is C^1-smooth (actually C^∞-smooth). Hence P is a null-set by Problem 3. This implies that every subset N of P is also a null-set.