Remarks for the solutions of Homework 2

I will present a detailed solution for Problem 1(b) and give some hints for the other problems.

Problem 1. (a) The intersection

$$\lambda(\mathcal{S}) := \bigcap\{\mathcal{A} : \mathcal{A} \text{ is a } \lambda\text{-system on } X \text{ that contains } \mathcal{S}\}$$

of all λ-systems that contain $\mathcal{S}$ is the smallest λ-system that contains $\mathcal{S}$.

(b) Suppose that $\mathcal{S}$ is a π-system on X and $\lambda(\mathcal{S})$ is the smallest λ-system that contains $\mathcal{S}$. We want to show that $\lambda(\mathcal{S})$ is a π-system. To see this, we first establish the following statement.

Claim 1. Let $S \in \mathcal{S}$ be arbitrary and define $\mathcal{A} := \{A \subseteq X : S \cap A \in \lambda(\mathcal{S})\}$. Then $\mathcal{A}$ is a λ-system that contains $\mathcal{S}$ and so $\lambda(\mathcal{S}) \subseteq \mathcal{A}$.

For the proof of Claim 1 first note that if $S' \in \mathcal{S}$ is arbitrary, then $S \cap S' \in \mathcal{S} \subseteq \lambda(\mathcal{S})$, because $\mathcal{S}$ is a π-system. Hence $S' \in \mathcal{A}$. It follows that $\mathcal{S} \subseteq \mathcal{A}$ as desired.

It remains to show that $\mathcal{A}$ is a λ-system. To see this, we verify conditions (i)–(iii) of a λ-system for $\mathcal{A}$.

(i) We have $X \in \mathcal{A}$, because $S \cap X = S \in \mathcal{S} \subseteq \lambda(\mathcal{S})$.

(ii) Suppose $A, B \in \mathcal{A}$ are arbitrary. Then $A \cap S \in \lambda(\mathcal{S})$ and $B \cap S \in \lambda(\mathcal{S})$ by definition of $\mathcal{A}$. Hence by condition (ii) for the λ-system $\lambda(\mathcal{S})$ we have

$$(A \cap S) \setminus (B \cap S) \in \lambda(\mathcal{S}).$$

This in turn implies that (we use the usual notation $N^c := X \setminus N$ for $N \subseteq X$)

$$\begin{aligned}(A \setminus B) \cap S &= (A \cap B^c) \cap S = (A \cap S) \cap B^c \\ &= (A \cap S) \cap (B^c \cup S^c) \qquad (\text{note that } (A \cap S) \cap S^c = \emptyset) \\ &= (A \cap S) \cap (B \cap S)^c \qquad (B^c \cup S^c = (B \cap S)^c \text{ by de Morgan}) \\ &= (A \cap S) \setminus (B \cap S) \in \lambda(\mathcal{S}).\end{aligned}$$

This implies $A \setminus B \in \mathcal{A}$ as desired.

(iii) Suppose $A_n \in \mathcal{A}$ for $n \in \mathbb{N}$ are arbitrary sets with $A_n \nearrow$. Then we have $A_n \cap S \in \lambda(\mathcal{S})$ for $n \in \mathbb{N}$. Moreover, for each $n \in \mathbb{N}$ we have $A_n \subseteq A_{n+1}$, and so $A_n \cap S \subseteq A_{n+1} \cap S$. Hence $A_n \cap S \nearrow$. If we apply condition (iii) for the λ-system $\lambda(\mathcal{S})$, then we see that

$$\left(\bigcup_{n \in \mathbb{N}} A_n\right) \cap S = \bigcup_{n \in \mathbb{N}} (A_n \cap S) \in \lambda(\mathcal{S}).$$

It follows that $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$ as desired.

We conclude that $\mathcal{A}$ is a λ-system that contains $\mathcal{S}$. In particular, $\lambda(\mathcal{S}) \subseteq \mathcal{A}$, because $\lambda(\mathcal{S})$ is the smallest λ-system that contains $\mathcal{S}$. Claim 1 follows.

Claim 2. We have $S \cap S' \in \lambda(\mathcal{S})$ for all $S \in \mathcal{S}$ and $S' \in \lambda(\mathcal{S})$.

This easily follows from Claim 1. Indeed, let $S \in \mathcal{S}$ and $S' \in \lambda(\mathcal{S})$ be arbitrary. Define $\mathcal{A}$ for the set S as in Claim 1. Then Claim 1 shows that $\lambda(\mathcal{S}) \subseteq \mathcal{A}$. In particular, $S' \in \mathcal{A}$. This means that $S \cap S' \in \lambda(\mathcal{S})$ by definition of $\mathcal{A}$.

Claim 3. Let $S \in \lambda(\mathcal{S})$ be arbitrary and define $\mathcal{A} := \{A \subseteq X : S \cap A \in \lambda(\mathcal{S})\}$. Then $\mathcal{A}$ is a λ-system that contains $\mathcal{S}$ and so $\lambda(\mathcal{S}) \subseteq \mathcal{A}$.

Note that while in Claim 1 we assumed $S \in \mathcal{S}$, here we make the weaker assumption $S \in \lambda(\mathcal{S})$.

The proof of Claim 3 is very similar to the proof of Claim 1. First note that if $S' \in \mathcal{S}$ is arbitrary, then $S \cap S' \in \lambda(\mathcal{S})$ by Claim 2 (note that there the roles of S and S' are reversed). Hence $S' \in \mathcal{A}$. It follows that $\mathcal{S} \subseteq \mathcal{A}$ as desired.

That $\mathcal{A}$ is a λ-system is proved in the same way as in the proof of Claim 1. Indeed, to see that property (i) of a λ-system is true for $\mathcal{A}$, we note that $X \in \mathcal{A}$, because $X \cap S = S \in \lambda(\mathcal{S})$ by the hypothesis in Claim 3. The proofs of property (ii) and (iii) are identical to the proofs in Claim 1. The arguments there were valid for an arbitrary set $S \subseteq X$. Claim 3 follows.

Now we can show that $\lambda(\mathcal{S})$ is a π-system. Indeed, let $S, S' \in \lambda(\mathcal{S})$ be arbitrary and define $\mathcal{A}$ for the set S as in Claim 3. Then Claim 3 shows that $\lambda(\mathcal{S}) \subseteq \mathcal{A}$. In particular, $S' \in \mathcal{A}$. This means that $S \cap S' \in \lambda(\mathcal{S})$ by definition of $\mathcal{A}$. The statement follows.

(c) By (b) the λ-system $\lambda(\mathcal{S})$ is also a π-system. Moreover, every λ-system that is also a π-system is a σ-algebra. Hence $\lambda(\mathcal{S})$ is a σ-algebra. The statement easily follows.

Problem 2. (a) Consider the family $\mathcal{A}$ of all sets $A \in \mathcal{F}$ such that $\mu(A) = \nu(A)$. Show that $\mathcal{A}$ is a λ-system that contains $\mathcal{S}$. Then $\sigma(\mathcal{S}) \subseteq \mathcal{A}$ by the π-λ-Theorem. This implies $\mu = \nu$.

For a counterexample consider $X = \{1, 2, 3, 4\}$, $\mathcal{F} = \mathcal{P}(X)$, $\mathcal{S} = \{\{1, 2\}, \{1, 3\}\}$, $\mu = \frac{1}{2}(\delta_1 + \delta_4)$, $\nu = \frac{1}{2}(\delta_2 + \delta_3)$, where δ_a denotes Dirac measure at $a \in X$. Note that $\sigma(\mathcal{S}) = \mathcal{P}(X)$.

(b) Apply (a) to the measures obtained by "restricting to" S_n (as in Prob. 3(a)). Then $\mu(A) = \nu(A)$ whenever $A \in \mathcal{F}$ and $A \subseteq S_n$. Then "make the sets S_n disjoint" and use countable additivity.

Problem 3. (a) Easy.

(b) Break $\mathbb{R}^n$ up into pairwise disjoint Borel sets F_n, $n \in \mathbb{N}$, with $\mu(F_n) < \infty$. Each of the "restricted" measures $\mu|F_n$ (defined as in (a)) is regular as follows from HW1, Prob. 2. Use this in combination with careful estimates.

Problem 4. One implication is trivial. For the other implication assume that $\mu(A) = 0 \Rightarrow \nu(A) = 0$. Then argue by contradiction: suppose there exists $\epsilon > 0$ ("bad" ϵ) such that no matter how small $\delta > 0$ is, there exists a set $A \in \mathcal{A}$ with $\mu(A) < \delta$, but $\nu(A) \geq \epsilon$. Use this for $\delta = 1/2^n$ and obtain sets A_n. Consider $A = \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k$.

For the counterexample consider counting measure ν be on $\mathbb{N}$, and the measure μ on $\mathbb{N}$ that assigns weight $1/2^n$ to n, i.e.,

$$\mu(A) = \sum_{n\in A} \frac{1}{2^n}$$

for $A \subseteq \mathbb{N}$.