Remarks for the solutions of Homework 1

I will present a detailed solution for Problem 1 (b) (including part (a)) and give some hints for the other problems.

Problem 1. Suppose X is a set and $\mathcal{A}$ is a σ-algebra on X with infinitely many elements. We want to show that then $\mathcal{A}$ contains uncountably many elements. The idea of the proof is to find an injective map from $\mathcal{P}(\mathbb{N})$ (the power set of $\mathbb{N}$) into $\mathcal{A}$. Since we know that $\mathcal{P}(\mathbb{N})$ is uncountable, the statement will easily follow from this. We now provide the details. We will establish three claims.

Claim 1. There exist sets $A_n \in \mathcal{A}$ for $n \in \mathbb{N}$ such that $A_{n+1} \subseteq A_n$ and $A_{n+1} \neq A_n$ for all $n \in \mathbb{N}$.

In other words, we claim that there exists a strictly descending sequence of sets in $\mathcal{A}$. In order to prove Claim 1, we define the desired sets A_n inductively. For this in turn it is useful to introduce the following concept: we call a set $A \subseteq X$ *rich* if $A \in \mathcal{A}$ and if A contains infinitely many (distinct) subsets that belong to $\mathcal{A}$.

Subclaim 1. If $A \subseteq X$ is rich, then there exists a rich set B with $B \subseteq A$ and $B \neq A$.

In other words, every rich set has a rich proper subset (as usual, a set N is called a *proper* subset of a set M if $N \subseteq M$, but $N \neq M$). To prove Subclaim 1, let $A \in \mathcal{A}$ be an arbitrary rich set. Then A has infinitely subsets that belong to $\mathcal{A}$. In particular, $\emptyset$ and A cannot be the only subsets of A that belong to $\mathcal{A}$. Hence there exists a set $E \subseteq A$ with $E \in \mathcal{A}$, $E \neq \emptyset$, and $E \neq A$. Then $F := A \setminus E$ also belongs to $\mathcal{A}$, because A and E belong to $\mathcal{A}$ and the σ-algebra $\mathcal{A}$ is closed under taking differences of sets in $\mathcal{A}$. Note that both $E, F \in \mathcal{A}$ are proper subsets of A. These sets satisfy $E \cup F = A$.

If E or F is a rich subset of A, then Subclaim 1 is true (choose $B = E$ or $B = F$).

We will show that the alternative, namely, that neither E nor F is a rich subset of A, leads to a contradiction (the subclaim then follows). In this case, both E and F contain only a finite number of sets in $\mathcal{A}$, say, the set E contains the sets $M_1, \dots, M_n \in \mathcal{A}$, where $n \in \mathbb{N}$, and F contains the sets $N_1, \dots, N_k \in \mathcal{A}$, where $k \in \mathbb{N}$, and there are no other sets in $\mathcal{A}$ contained in E or F (note that $n, k \geq 1$, because $\emptyset \in \mathcal{A}$, and $\emptyset \subseteq E$, $\emptyset \subseteq F$).

Now consider an arbitrary set $D \in \mathcal{A}$ with $D \subseteq A$. Then $D \cap E \in \mathcal{A}$, because $D, E \in \mathcal{A}$ and, as a σ-algebra, $\mathcal{A}$ is closed under taking finite intersections of sets in $\mathcal{A}$. Since $D \cap E \subseteq E$, there exists some $i \in \{1, \dots, n\}$ such that $D \cap E = M_i$. The same reasoning shows that $D \cap F = N_j$ for some $j \in \{1, \dots, k\}$. Then

$$D = D \cap A = D \cap (E \cup F) = (D \cap E) \cap (D \cap F) = M_i \cup N_j.$$

In other words, every subset D of A that belongs to $\mathcal{A}$ can be represented in the form

$$D = M_i \cup N_j$$

with $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, k\}$. So there at most $n \cdot k$ such sets, i.e., a finite number of these sets. This contradicts the fact that A is rich. We obtain a contradiction, as desired. Subclaim 1 follows.

Subclaim 2. There exist rich sets A_n for $n \in \mathbb{N}$ such that A_{n+1} is a proper subset of A_n for each $n \in \mathbb{N}$.

We define the sets A_n inductively. Let $A_1 := X$. This is a rich set, because $A_1 = X \in \mathcal{A}$ by definition of a σ-algebra (on X) and $A_1 = X$ has infinitely many subsets that belong to $\mathcal{A}$ (since $\mathcal{A}$ is infinite).

Suppose for $n \in \mathbb{N}$ the rich sets $A_1, \dots, A_n$ have been chosen so that each set is a proper subset of the previous one in the list. Then by Subclaim 1, we can find a rich proper subset B of A_n. If we define $A_{n+1} := B$, then A_{n+1} is a rich proper subset of A_n as desired.

By the induction principle, we get rich sets A_n for all $n \in \mathbb{N}$ with the desired inclusion properties.

Claim 1 immediately follows from Subclaim 2, because in Claim 1 we can choose the same sets as in Subclaim 2, because by definition every rich set belongs to $\mathcal{A}$.

Claim 2. There exist sets $B_n \in \mathcal{A}$ for $n \in \mathbb{N}$ such that $B_n \neq \emptyset$ and $B_n \cap B_k = \emptyset$ for all $n, k \in \mathbb{N}$, $n \neq k$.

In other words, $\mathcal{A}$ contains an infinite sequence of pairwise disjoint non-empty sets. This easily follows from Claim 1. Indeed, if the sets $A_n \in \mathcal{A}$ are as in Claim 1, then we define $B_n := A_n \setminus A_{n+1}$ for $n \in \mathbb{N}$. Then B_n is the difference of the sets $A_n, A_{n+1} \in \mathcal{A}$, and so $B_n \in \mathcal{A}$, because $\mathcal{A}$ is a σ-algebra. Since A_{n+1} is a proper subset of A_n, we have $B_n = A_n \setminus A_{n+1} \neq \emptyset$ for all $n \in \mathbb{N}$.

To see that the sets B_n, $n \in \mathbb{N}$, are pairwise disjoint, let $n, k \in \mathbb{N}$ with $n \neq k$ be arbitrary. Since the roles of n and k are symmetric, we may assume that $k > n$. Then $k \geq n + 1$ and so $A_k \subseteq A_{n+1}$. This in turn implies

$$B_n \cap B_k \subseteq (A_n \setminus A_{n+1}) \cap A_k \subseteq (A_n \setminus A_{n+1}) \cap A_{n+1} = \emptyset.$$

Hence $B_n \cap B_k = \emptyset$, as desired. Claim 2 follows.

Claim 3. If the sets B_n, $n \in \mathbb{N}$, are as in Claim 2, then the map $\varphi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{A}$ defined as

$$\varphi(I) := \bigcup_{n \in I} B_n \tag{1}$$

for $I \subseteq \mathbb{N}$ is injective.

Note that $\mathcal{P}(\mathbb{N})$ is the power set of $\mathbb{N}$, i.e., the set of all subsets I of $\mathbb{N}$. So the definition in (1) makes sense, because it assigns to each element in $\mathcal{P}(\mathbb{N})$, i.e., each subset I of $\mathbb{N}$ the subset $\varphi(I)$ of X. Actually, $\varphi(I)$ lies in $\mathcal{A}$, because every subset I of $\mathbb{N}$ is countable and a countable union of the sets $B_n \in \mathcal{A}$ stays in $\mathcal{A}$, because $\mathcal{A}$ is a σ-algebra. Note that if $I = \emptyset$, then $\varphi(\emptyset)$ is an empty union. By standard convention, this is interpreted as the empty set; so $\varphi(\emptyset) = \emptyset \in \mathcal{A}$. These considerations show that (1) indeed defines a map $\varphi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{A}$ as stated in Claim 3. After we convinced ourselves that Claim 3 is a meaningful statement, we will now prove it.

So let $I, J \subseteq \mathbb{N}$ be arbitrary subsets of $\mathbb{N}$ with $I \neq J$. To show injectivity of φ, we have to show that $\varphi(I) \neq \varphi(J)$. Since $I \neq J$, we cannot have both $I \subseteq J$ and $J \subseteq I$ (otherwise, $I = J$). Since the roles of I and J are symmetric, let us assume that $I \not\subseteq J$. Then there exists $k \in I$ with $k \notin J$; but then

$$\emptyset \neq B_k \subseteq \bigcup_{n \in I} B_n = \varphi(I),$$

and

$$\varphi(J) \cap B_k = \left(\bigcup_{n \in J} B_n \right) \cap B_k = \bigcup_{n \in J} (B_n \cap B_k) = \emptyset,$$

because $k \notin J$ and the sets B_n, $n \in \mathbb{N}$, are non-empty and pairwise disjoint. So $B_k \neq \emptyset$ is a subset of $\varphi(I)$, but not of $\varphi(J)$. This shows that $\varphi(I) \neq \varphi(J)$ as desired. Claim 3 follows.

We can now prove that $\mathcal{A}$ is uncountable. Essentially, this immediately follows from Claim 3. Here are the details. We argue by contradiction and assume that $\mathcal{A}$ is countable. Then every subset of $\mathcal{A}$ is countable; in particular, the image set $\varphi(\mathcal{P}(\mathbb{N})) \subseteq \mathcal{A}$ of the map φ in Claim 3 is countable. Since $\varphi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{A}$ is injective, this map can be considered as a bijection $\varphi\colon \mathcal{P}(\mathbb{N}) \to \varphi(\mathcal{P}(\mathbb{N}))$ onto its image (this map is injective and surjective). Hence $\mathcal{P}(\mathbb{N})$ is countable; but we know that this is not the case (this is a well-kown fact discussed in the TA session). We get a contradiction, showing that $\mathcal{A}$ is indeed unaccountable.

Problem 2. (a) Easy, if you remember that every closed set in $\mathbb{R}^n$ admits an "exhaustion" by compact sets.

(b) Let $\epsilon > 0$. Choose compact K_n and open U_n such that

$$K_n \subseteq B_n := A_n \setminus A_{n-1} \subseteq U_n$$

and $\mu(U_n \setminus K_n) < \epsilon/2^n$ for $n \in \mathbb{N}$.

For $A = \bigcup_{n \in \mathbb{N}} A_n$ choose $K = \bigcup_{n=1}^N K_n$ with N large and $U = \bigcup_{n \in \mathbb{N}} U_n$. Estimate carefully using that $\lim_{n \to \infty} \sum_{i=n}^\infty \mu(B_i) = 0$.

(c) Show that the σ-algebra $\mathcal{A}$ contains each open set U (follows from the fact that U admits a "compact exhaustion"). Hence $\mathcal{A}$ contains each Borel set.

Problem 3. (a) Observe that

$$A = \bigcap_{n \in \mathbb{N}} \bigcup_{k \geq n} A_k.$$

(b) Use (a) and play around with tails of infinite series and limits.

Problem 4. The basic idea is to show that

$$\mathcal{A} = \{B \subseteq \mathbb{R} : f(B) \text{ is Borel}\}$$

is a σ-algebra on $\mathbb{R}$ that contains all intervals. Then $\mathcal{A}$ contains the Borel σ-algebra on $\mathbb{R}$.

For this one shows that if $I \subseteq \mathbb{R}$ is an interval (i.e., a connected subset of $\mathbb{R}$), then $\mathbb{R} \setminus f(I)$ has no connected components that are singleton sets. This implies the connected components of the set $\mathbb{R} \setminus f(I)$ form a family of pairwise disjoint non-degenerate intervals. This is necessarily a countable family. It follows that $\mathbb{R} \setminus f(I)$, and hence also $f(I)$ is a Borel set.

To show that $\mathcal{A}$ is a σ-algebra, one needs to verify (among other things) that if $A \in \mathcal{A}$, then $\mathbb{R} \setminus A \in \mathcal{A}$. For this one shows that $f(\mathbb{R}) \setminus f(A)$ (which is a Borel set) differs from $f(\mathbb{R} \setminus A) \supseteq f(\mathbb{R}) \setminus f(A)$ by an at most countable set.