Prop. \( \nu \) is well-defined and a premeasure on \( \mathcal{U} \).

Proof. Suppose \( M \in \mathcal{U} \) and
\[
M = R_1 \cup \ldots \cup R_k = S_1 \cup \ldots \cup S_l
\]
for disjoint unions of \( h \)-rectangles \( R_i \) and \( S_j \).

Then by Basic Laws,
\[
A = \sum_{i=1}^{k} \mu(R_i) = B = \sum_{j=1}^{l} \nu(S_j)
\]
Similarly, \( B = A \) and so \( A = B \).

i.e., \( \nu(M) \) is well-defined.

\( \nu \) is a premeasure on \( \mathcal{U} \):
\[
\nu(\emptyset) = \left| \emptyset \times \ldots \times \emptyset \right| = \ell(\emptyset) \cdots \ell(\emptyset) = 0.
\]

Suppose \( A \in \mathcal{U} \), \( n \in \mathbb{N} \), and \( \nu \) is a premeasure disjoint and \( U A_n \in \mathcal{U} \).

\( \forall n \in \mathbb{N} \)
\[
\text{WTS } \nu(U A_n) = \sum_{n=1}^{\infty} \nu(A_n).
\]

We have disjoint unions of \( h \)-rectangles:
\[
A_n = R_{n1} \cup \ldots \cup R_{nk_n}, \quad n \in \mathbb{N}
\]
\[
U A_n = S_1 \cup \ldots \cup S_l, \quad n \in \mathbb{N}
\]

Since the \( A_n \)'s are pairwise disjoint
the \( h \)-rectangles \( R_{ni} \), \( n \in \mathbb{N}, \quad i = 1, \ldots, k_n
\]
are also pairwise disjoint.
(3) 2) Each h-rect. \( R \) is a Borel set and the h-rect. \( R \) generate the 
\( \sigma \)-alg. \( B^h = B^h \) of Borel sets on 
\( \mathbb{R}^n \).

In particular, each Borel set is 
(Lebesgue) measurable (by Carathéodory's 
Ext. Thm.).

So each h-rect. \( R \) is measurable and
\[ \lambda (R) = \lambda^h (R) = \nu (R) = |R|. \]

3) Lebesgue measure is complete (each 
measure induced by an outer measure is).

4) A rectangle \( R \) on \( \mathbb{R}^n \) is a compact 
set of 11. U. form
\[ R = [a_1, b_1] \times \ldots \times [a_n, b_n], \]
where \( a_i, b_i \in \mathbb{R} \), \( a_i \leq b_i \), \( \forall i = 1, \ldots, n \).

A rectangle is measurable (it is Borel) and
\[ \lambda (R) = (b_1 - a_1) \ldots (b_n - a_n) = |R| \]
(Idea of proof: there are h-rect. \( S, T \)
with \( S \subseteq R \subseteq T \) and \( |T| - |S| \) small).

In particular,
\[ \lambda \left( [0,1]^n \right) = 1. \]

4. \( M \subseteq \mathbb{R}^n \), then
\[ \lambda^h (M) = \inf \left\{ \sum_{i=1}^{\infty} \lambda (R_i) : R_i \subseteq M, \text{ rect. and } \right\} \]
\[ M = \bigcup R_i, \quad \text{and} \]
\( M \subseteq \bigcup R_i \).
Moreover,
\[
0 
\leq \bigcup_{i=1}^{k} R_{i} = U A \bigcup_{n \in \mathbb{N}} = S \cup \ldots \cup S
\]

5.5 Basic Lemma
\[
\sum_{n=1}^{N} |R_{n}| = \sum_{j=1}^{\infty} |S_{j}| = \nu (U A_{n})
\]

Det. (Lebesgue measure on \( \mathbb{R}^{n} \))
The outer measure induced by the premeasure \( \nu \) on the algebra generated by \( h \)-rectangles is called Lebesgue outer measure denoted by \( m_{n}^{*} \) (or \( \mathcal{A} = \mathcal{A}_{n}^{*} \) if not understood, or \( \mathcal{A}_{\infty} \), etc).

The measure induced by \( m_{n}^{*} \) (according to Carathéodory's Theorem) is called Lebesgue measure, denoted by \( m_{n} \), and the \( m_{n}^{*} \)-measurable sets Lebesgue measurable (or just measurable).

Rem. 1) If \( M \subseteq \mathbb{R}^{n} \), then
\[
m_{n}^{*} (M) = \inf \left\{ \sum_{n=1}^{\infty} \nu (A_{n}) : A_{n} \in \text{alg. gen. by } h \text{-rect.}, M \subseteq \bigcup_{n \in \mathbb{N}} A_{n} \right\}
\]

\[
= \inf \left\{ \sum_{i=1}^{\infty} |R_{i}| : R_{i} \text{ } h \text{-rect.}, M \subseteq \bigcup_{n \in \mathbb{N}} R_{i} \right\}
\]

\[
\text{content of } h \text{-rect.}
\]
Theorem. Let \( M \subseteq \mathbb{R}^n \). TFAE:

1) \( \lambda^*(M) = 0 \) or null set,
2) \( M \) is a set of measure zero,
3) \( \lambda(M) = 0 \)
4) for all \( \varepsilon > 0 \) there exist rectangles \( R_n \) such that \( M \subseteq \bigcup_{n=1}^{\infty} R_n \) and \( \sum_{n=1}^{\infty} \lambda(R_n) < \varepsilon \).
5) there exists a Borel set \( B \subseteq \mathbb{R}^n \) with \( M \subseteq B \) and \( \lambda(B) = 0 \).

(Homework!)

Corollary. A set \( M \subseteq \mathbb{R}^n \) is (Lebesgue) measurable if and only if there exists a Borel set \( B \subseteq \mathbb{R}^n \) and a null set \( N \subseteq \mathbb{R}^n \) such that \( M = B \cup N \).

Proof. Consider that Borel sets and null sets are measurable, and unions of measurable sets are measurable.

Suppose, in addition, that \( \lambda(M) = \lambda^*(M) \). Then for each \( k \in \mathbb{N} \), there exists a Borel set \( B_k \) and null set \( N_k \) such that

\[
\sum_{i=1}^{\infty} \lambda(R_i) < \lambda(M) + \frac{1}{k}.
\]

Then \( \lambda(M) = \lambda(B_k) \leq \sum_{i=1}^{\infty} \lambda(R_i) \), countable subadditivity.
(3.3) Let \( A \subset B_k \) for each \( k \in \mathbb{N} \).

Then \( M \subset A \subset B_k \) for each \( k \in \mathbb{N} \), so
\[
\lambda(M) \leq \lambda(A) \leq \lambda(B_k) = \lambda(M) + \frac{1}{k}.
\]

Letting \( k \to \infty \), we conclude
\[
\lambda(M) = \lambda(A).
\]

So \( \lambda(A \setminus M) = 0 \).

Now by prev. thm. there ex. a
Borel set \( C \) with \( A \setminus M \subset C \) and
\[
\lambda(C) = 0.
\]

Then \( M = (A \setminus C) \cup (M \cap C) \) Borel \( \subset C \) so well def.

For the general case, write \( M = \bigcup_{n=1}^{\infty} M_n \) where \( M_n \) measurable
and \( \lambda(M_n) < \infty \).

(details left as exercise)

\[\square\]

**Thm.** Lebesgue measure on \( \mathbb{R}^n \) is the
unique measure \( \lambda \) s.t.

(i) is defined on the \( \mathcal{B} \)-algebra of all
(Lebesgue)measurable sets,

(ii) is translation-invariant:
\[
\lambda(M) = \lambda(t + M) \quad \text{for all } t \in \mathbb{R}^n,
\]

(iii) \( \lambda([0,1]^n) = 1 \), \( t + M = \{ t + m : m \in M \} \)

**Proof:** Lebesgue measure \( \lambda \) has the
Properties (i) + (iii) clear.

(ii): If \( R \& R^{*} \) n-rect., then
54. \( t + R \) also \( h \)-closed and \( l = |t + R| \); this implies 
\[
\lambda^+(M) = \lambda^+(t + M)
\]
for all \( M \in \mathbb{R} \), and 
\[
\lambda^+(M) \in \mathbb{R}^+ \text{ measurable, (i.e., Lebesgue measurable)}; \quad \lambda^+(t + M) \text{ - mass}.
\]
So if \( M \in \mathbb{R}^n \) measurable, then \( t + M \) non-
and 
\[
\lambda^+(t + M) = \lambda^+(t + M) = \lambda^+(M) = \lambda(M).
\]
For the uniqueness statement suppose \( m \) is a translation-invariant Borel measure on \( \mathbb{R}^n \) with \( C_0 = \mu \left( \left[ 0, 1 \right]^n \right) < 2^n \).

Claim \( \lambda (B) = C_0 \lambda (B) \) for each Borel closed \( B \subseteq \mathbb{R}^n \).

I. Step. For each \( h \)-cube \( Q = \left[ a_i, a_i + b_i / k \right]^n \), we have 
\[
\lambda (Q) = \frac{1}{k^n} C_0 = C_0 \lambda (Q) \]
indeed.

By translation-invariance all \( h \)-cubes 
with some side-length have the same measure. 
Now id \( Q = (0, 1/k]^n \). Then 
\[
Q^* = (0, 1]^n = \bigcup \left( \frac{1}{k} + Q \right)
\]
ed \( 0, 1/k - 1^2 \) \( \lambda \)-almost join with \( Q \) by \( h \)-cubes of side-length \( \frac{1}{k} \).

Hence \( \lambda (Q^*) = k^n \lambda (Q) \) so 
\[
\mu (Q) = \frac{1}{k^n} \mu (Q^*) = \frac{1}{k^n} C_0 = C_0 \lambda (Q). \]
II. Step: Show claim for $h$ - rectangles with rational side lengths. (based on I. Step)

III. Show claim for arbitrary rectangles (based on II. Step and limiting argument).

IV. Step: (Exercise!)

Now consider the measures $\mu$ and $\mu^*$ defined on Borel $\sigma$-rings $\mathcal{B}$ in $\mathbb{R}^n$.

III. Step: The measures agree on the $\sigma$-system of all rectangles.

Hence (follows from HW2, Prob. 2a+6)) the measures agree on all Borel sets.

The claim follows.

Now let $\mu$ be a measure with prop. (i) + (ii) $\Rightarrow$ (iii).

Then $C_0 = \mu \left( [0,1]^n \right) \leq \mu \left( \mathbb{R}^n \right) = 1$.

So by Claim,

$$\mu (B) = C_0 \cdot \lambda (B) \quad \text{for all Borel sets } B \in \mathbb{R}^n.$$

Hence $1 = \mu \left( \mathbb{R}^n \right) = C_0 \cdot \lambda (\mathbb{R}^n) = C_0 \cdot 1 \Rightarrow C_0 = 1$.

and

$$\mu (B) = \lambda (B) \quad \text{for all Borel sets } B \in \mathbb{R}^n.$$

This implies (exercise!)

$$\mu (N) = \lambda (N) = 0 \quad \text{for all } \lambda \text{- null sets } N \in \mathbb{R}^n.$$

If $M \in \mathbb{R}^n$ is measurable, then $M = B \cup N$, where $B \in \mathbb{R}^n$ Borel and $N \in \mathbb{R}^n$ $\lambda$-null set.

Hence

$$\mu (M) = \mu (B) + \mu (N) = \lambda (B) \leq \lambda (M).$$
Similarly, \( \lambda(M) \leq \mu(M) \) for all measurable \( M \subset \mathbb{R}^n \).

Other properties of Lebesgue measure:

1) \( \lambda \) is regular, i.e.,
   for each measurable \( M \subset \mathbb{R}^n \) we have
   \[ \lambda(M) = \inf \{ \lambda(A) : A \supset M, \text{compact} \} \]
   \[ \lambda(M) = \sup \{ \lambda(U) : U \subset M, \text{open} \} \]
   (inner regularity)
   (outer regularity)
   (follows from HW 2, Prob. 03).

2) Behavior under linear maps:
   Let \( L : \mathbb{R}^n \to \mathbb{R}^n \) be a linear map.
   If \( M \subset \mathbb{R}^n \) is measurable, then \( \lambda(M) \)
   is also measurable, and
   \[ \lambda(L(M)) = |\det(L)| \cdot \lambda(M) \]
   (homework!).
   In particular, if \( L \) is an orthogonal transformation related to the \( n \)
   \[ \lambda(L(M)) = \lambda(M) \]
   "Lebesgue measure is rotation-invariant."

Not every subset \( M \subset \mathbb{R}^n \) is (Lebesgue) measurable.

Ex. for \( n = 1 \):

We define an equivalence relation on \([0,1] : \) For \( x, y \in [0,1] : x \sim y \iff y - x \in \mathbb{Q} \).
As usual, we denote the equivalence class of \( x \) as
\[
[x] = \{ y \in [0,1]: y - x \}
\]
Then
\[
[x] = [y] \text{ or } [x] \cap [y] = \emptyset \text{ for any } x, y.
\]
Now choose a representative \( x \) from each equivalence class (axiom of choice), call the set \( E \) of the representatives.

Then

i) \( E \subseteq [0,1] \)

ii) \( (5 \cdot E) \cap (r + E) = \emptyset \) for \( r, s \in \mathbb{Q} \), suppose not ! Then \( r \neq s \).

\[ x - y = r - s \in \mathbb{Q} \text{, and } x \sim y, \]
but \( E \) contains precisely one element from each equivalence class. Contradiction.

iii) \( \lambda([0,1]) = \lambda(U(r + E)) = \lambda([-1/2, r + 1/2]) \), \( r \in [-1/2, 1/2] \), \( \emptyset \)

Indeed if \( y \in [0,1] \) is outside the \( r \), then \( y = x + r \in r + E, \) \( x \in E \) s.t. \( y - x = r \in \emptyset \cap [-1/2, 1/2] \).

Thus \( y \in [0,1] \).

Claim The set \( E \) is not Lebesgue measurable.

Proof by contradiction. Suppose not !

Then \( E \) and hence each of its translates \( r + E \), \( r \in \emptyset \) is measurable.

Then by (ii) and (iii):

\[
1 = \lambda([0,1]) = \lambda(U(r + E)), \quad r \in [-1/2, 1/2] \cap \emptyset
\]
\[ \sum_{t \in \mathbb{R} \setminus \emptyset} \lambda (t + E) \quad \text{(Poincaré Disjunction)} \]

\[ \sum_{t \in \mathbb{R} \setminus \emptyset} \lambda (E) \quad \text{(Translation Invariance)} \]

So \( \lambda (E) > 0 \) and \( \sum t = +\infty \)

On the other hand:

\[ +\omega = \sum_{t \in \mathbb{R} \setminus \emptyset} \lambda (E) = \lambda (U (t + E)) \]

\[ \leq \lambda ([\mathbb{R}]) = 3 \quad \text{(Contradiction)} \]

**Measurable Functions**

Let \((\mathbb{X}, \mathcal{A})\) and \((\mathbb{Y}, \mathcal{B})\) be measure spaces.

Then a map \( f : \mathbb{X} \to \mathbb{Y} \) is called \( (\mathcal{A}, \mathcal{B}) \)-measurable or just measurable (with \( \mathcal{A}, \mathcal{B} \) understood) if

\[ f^{-1} (B) = \{ x \in \mathbb{X} : f(x) \in B \} \in \mathcal{A} \quad \text{for each} \quad B \in \mathcal{B} \]

(projections of measurable sets are measurable).

**Facts**

1. If \( \mathcal{B} = \sigma (Y) \) (i.e., \( \mathcal{B} \) is generated by \( Y \)), then
$f: (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$ is measurable if and only if $f^{-1}(B) \in \mathcal{A}$ for each $B \in \mathcal{B}$.

Proof: \( \Rightarrow \) Claim.

Let \( \mathcal{F} = \{B \subseteq X : f^{-1}(B) \in \mathcal{A}\} \).

Then \( Y \in \mathcal{F} \) by assumption.

Claim: \( \mathcal{F} \) is a \( \sigma \)-algebra on \( Y \).

i) \( \emptyset \in \mathcal{F} \):

\[ f^{-1}(\emptyset) = \emptyset \in \mathcal{A}, \quad f^{-1}(Y) = X \in \mathcal{A}. \]

ii) Let \( B \in \mathcal{F} \). Then

\[ f^{-1}(B^c) = f^{-1}(X \setminus B) = X \setminus f^{-1}(B) \in \mathcal{A}. \]

So, \( B^c \in \mathcal{F} \).

iii) Suppose \( B_n \in \mathcal{F} \) for \( n \in \mathbb{N} \). Then

\[ f^{-1}\left( \bigcup_{n=1}^{\infty} B_n \right) = \bigcup_{n=1}^{\infty} f^{-1}(B_n) \in \mathcal{A}. \]

So, \( \bigcup_{n=1}^{\infty} B_n \in \mathcal{F} \).

By (i) - (iii), \( \mathcal{F} \) is a \( \sigma \)-algebra.

2) If \( f: (X, \mathcal{A}) \rightarrow (Y, \mathcal{B}), \ g: (Y, \mathcal{B}) \rightarrow (Z, \mathcal{C}) \) are measurable, then

\[ g \circ f: (X, \mathcal{A}) \rightarrow (Z, \mathcal{C}) \] is measurable (compositions of measurable maps are measurable).
40. Proof: If $C \in \mathcal{C}$, then $g^{-1}(C) \in \mathcal{B}$.
So $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C)) \in \mathcal{A}$. \hfill \Box

3) Let $X$, $Y$ be top. spaces equipped with their Borel $\sigma$-algebras, and $f : X \to Y$ continuous. Then $f$ is measurable (Continuous functions ("Borel measurable") are Borel measurable).

Proof: Let $O_X$ and $O_Y$ be the family of open sets in $X$ and $Y$. Then
\[ f^{-1}(O_Y) = \bigcup O_X. \]

Then $f^{-1}(O) \cup \{ \emptyset \}$ is in $\mathcal{B}_X$ for $O \in \mathcal{B}_Y$. Hence $f$ is measurable by Fact 1. \hfill \Box

The extended real line
\[ \overline{\mathbb{R}} = \mathbb{R} \cup \{ +\infty, -\infty \}. \]

Order: $\infty \geq a \geq b$ for all $a \in \mathbb{R}$, $a \leq -\infty$.

Algebraic operations:
\[ a + (\pm \infty) = +\infty \quad \text{for all} \quad a \in \mathbb{R} \setminus \{0\} \]
$\pm \infty = 0$, e.g.
$+\infty + (-\infty)$ undefined, e.g.

If $M \in \overline{\mathbb{R}}$, then $\inf M$, $\sup M \in \overline{\mathbb{R}}$
always defined.
41. Topology:

Let \( U \subseteq \mathbb{R} \) be open. If every point \( p \in U \) has a neighborhood \( N \) that belongs to \( U \), i.e.,

\[
p \in \mathbb{R}, \quad N = (p - \delta, p + \delta) \subseteq U \quad \text{for some } \delta > 0
\]

\( p = +\infty \), \( N = (a, +\infty] \subseteq U \quad \text{for some } a \in \mathbb{R} \)

\( p = -\infty \), \( N = [-\infty, a) \subseteq U \quad \text{for some } a \in \mathbb{R} \).

This defines a topology on \( \mathbb{R} \).

Note: \( \mathbb{R} \) is homeomorphic to \([-1, 1] \cup [-1, 1]

\[ y : \mathbb{R} \rightarrow [-1, 1] \cup [-1, 1] \]

\[ x \rightarrow \begin{cases} \frac{2}{\pi} \arctan x & x \in \mathbb{R} \\ 0 & x = \pm \infty \end{cases} \]

is a homeomorphism.

Borel sets: \( B \subseteq \mathbb{R} \) is Borel in \( \mathbb{R} \)

if and only if \( B \cap \mathbb{R} \) is Borel in \( \mathbb{R} \) (exercise!)

A function on a space \( X \) is \( \mathbb{R} \)-valued if and only if the target

\( \mathbb{R} \)

is accordingly the \( \mathbb{R} \)-valued function on \( X \).

We always equip the target \( X \) with

the Borel \( T_1 \)-topology on \( X \).
If $X$ carries a $\sigma$-algebra $\mathcal{A}$, then a function $f : X \to \mathbb{E}$ is measurable if $f^{-1}(M) \in \mathcal{A}$ for each $M \in \mathcal{B}_\mathbb{E}$.

Note: Often $X = \mathbb{R}$. Then there are two natural $\sigma$-algebras on $\mathbb{R}$:
- The Borel $\sigma$-algebra $\mathcal{B} = \mathcal{B}(\mathbb{R})$, and
- the $\sigma$-algebra generated by all Lebesgue measurable sets.

A function $f : \mathbb{R} \to \mathbb{E}$ is called Lebesgue measurable if just measurable if
\[ f^{-1}(E) \in \mathcal{A}, \quad E \in \mathcal{B}_\mathbb{E} \]
is measurable, and
Borel measurable if
\[ f^{-1}(B) \in \mathcal{B}_\mathbb{E}, \quad B \in \mathcal{B}(\mathbb{R}) \]
is measurable.

Borel measurable $\Rightarrow$ (Lebesgue) measurable.

Note: A function $f : \mathbb{R} \to \mathbb{R}$ is measurable according to this ideal if $f^{-1}(E) \in \mathcal{A}$, $E \in \mathcal{B}_\mathbb{R}$ is measurable.

So, sources and targets carry different $\sigma$-algebras.

If $f, g : \mathbb{R} \to \mathbb{R}$ are measurable, then $g \circ f$ need not be measurable!
But a measurable function $f: \mathbb{R} \to \mathbb{R}$ followed by a Borel measurable function $g: \mathbb{R} \to \mathbb{R}$ is measurable.

A measurable function $f: (\mathbb{R}, \mathcal{B}) \to (\mathbb{R}, \mathcal{B})$.

**Lemma.** $(X, \mathcal{A})$ measure space.

1. $f: X \to \mathbb{R}$ is measurable if and only if $f^{-1}([a, b]) \in \mathcal{A}$ for each $a \leq b$.

2. If $u, v: X \to \mathbb{R}$ are measurable and $F: \mathbb{R}^2 \to Z$ is continuous, then $h: (X, \mathcal{A}) \to (Z, \mathcal{B}_Z)$, $h(x) = F(u(x), v(x))$, $x \in X$, is measurable.

**Proof.** (i) $\to$ clear because $[a, b] = \mathbb{R}$ open.

$\leftarrow$ True because the intervals $(a, b)$ $a \leq b$ generate $\mathcal{B}_{\mathbb{R}}$ (exercise). (ii) Define $f: X \to \mathbb{R}^2$, $x \mapsto f(x) = (u(x), v(x))$.

Claim $f: (X, \mathcal{A}) \to (\mathbb{R}^2, \mathcal{B})$ is measurable.

Enough to check $f^{-1}(R) \in \mathcal{A}$ for all rectangles $R = [a, b] \times [c, d]$. 