Measure theory =

math. theory that puts concept of volume and integrals on a sound basis.

\( \mathbb{R}^n \)

volume of sets (area in \( \mathbb{R}^2 \), length in \( \mathbb{R}^1 \))
system of sets

not necessarily measurable

a volume

\( \mathcal{X} \)

measure

\( \sigma \)-algebra \( \mathcal{A} \)

measurable subsets of \( \mathcal{X} \).

**Def. (\( \sigma \)-algebras)**

\( \mathcal{X} \) set, \( \mathcal{A} \) family of subsets of \( \mathcal{X} \).

Then \( \mathcal{A} \) is called a \( \sigma \)-algebra if the following axioms are true:

i) \( \emptyset, \mathcal{X} \in \mathcal{A} \)

ii) if \( A \in \mathcal{A} \), the \( A^c = \mathcal{X} \setminus A \in \mathcal{A} \)

iii) if \( A_1, A_2, \ldots, A_n \in \mathcal{A} \), complement of \( A \) (i.e. \( \mathcal{X} \))

\( \cap A_n \in \mathcal{A} \)

\( \cup (A_n \cap \mathcal{X}) \in \mathcal{A} \) (\( \mathcal{A} \) is closed under taking countable unions)

**Note:** 1) If \( \mathcal{A} \) is a \( \sigma \)-algebra, then

\( \mathcal{A} \) is closed under countable intersections as well.

i) \( A \in \mathcal{A} \) for all \( n \in \mathbb{N} \),

then \( \cap A_n \in \mathcal{A} \).
2) **Proof:** \( A^c \in \mathcal{A} \) by ii), so
\[
\bigcup_{n \in \mathbb{N}} A^c_n \in \mathcal{A} \text{ by i), so }
\bigcap_{n \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} A^c_n = \left( \bigcup_{n \in \mathbb{N}} A^c_n \right)^c \in \mathcal{A}, \quad \Box
\]

So, a \( \mathcal{F} \)-algebra is closed under all countable set-theoretic operations.

**Ex. 1)** \( \mathcal{P}(X) \) **power set of** \( X \)
\[
= \{ \text{set of all subsets of } X \}
\]
\( = \) a \( \mathcal{F} \)-algebra on \( X \).

2) **Let** \( A \)
\[
\mathcal{A} = \{ A \subseteq X : \text{A countable or } X \setminus A \text{ countable} \}
\]
Then \( \mathcal{A} \) is a \( \mathcal{F} \)-algebra on \( X \) (exercise!)

3) If \( A_i, i \in I \) is a family of \( \mathcal{F} \)-algebra on \( X \), then
\[
\mathcal{A} : = \bigcap_{i \in I} A_i = \{ A \subseteq X : A \in A_i \text{ for all } i \in I \}
\]
is also a \( \mathcal{F} \)-algebra on \( X \).
**Proof:** Straightforward. \text{for ex. prop. (iii)}.

If \( \Lambda \subseteq \mathcal{A} \) for \( \Lambda \subseteq \mathbb{N} \). Then \( A\in \mathcal{A} \) for all \( i \in \mathbb{N} \); so \( \bigcup_{i \in \mathbb{N}} A_i \in \mathcal{A} \) for all \( i \in \mathbb{N} \); so \( \bigcup_{i \in \mathbb{N}} A_i \in \mathcal{A} \).
4) If \( Y \subseteq P(X) \) is any family of subsets of \( X \), then there exists a smallest \( \sigma \)-algebra on \( X \) that contains \( Y \), denoted \( \sigma(Y) \), the \( \sigma \)-algebra generated by \( Y \); more precisely:

1) \( \sigma(Y) \) is a \( \sigma \)-algebra, with \( Y \subseteq \sigma(Y) \);
2) if \( \mathcal{A} \) is any \( \sigma \)-algebra, with \( Y \subseteq \mathcal{A} \), then \( \sigma(Y) \subseteq \mathcal{A} \);
3) \( \sigma(Y) \) is uniquely determined based on 1) + 2).

Proof:

Define \( \sigma(Y) = \bigcap \mathcal{A} \) \( \mathcal{A} \) is a \( \sigma \)-algebra, with \( Y \subseteq \mathcal{A} \).

Note: \( Y \subseteq P(X) \) is an intersection of non-empty subsets of \( P(X) \). This is a \( \sigma \)-algebra with \( Y \subseteq \sigma(Y) \). If \( \mathcal{A} \) is another \( \sigma \)-algebra, with \( Y \subseteq \mathcal{A} \), then \( \sigma(Y) \subseteq \mathcal{A} \) by def. of \( \sigma(Y) \). Finally, if \( \mathcal{A} \) is another \( \sigma \)-algebra, with \( \mathcal{A} \subseteq \mathcal{A} \), then \( \sigma(Y) \subseteq \mathcal{A} \) and \( \mathcal{A} \subseteq \mathcal{A} \), so \( \mathcal{A} \subseteq \sigma(Y) \).

5) The Borel \( \sigma \)-algebra on a topological space.

Let \( (X, \mathcal{O}) \) be a topological space, i.e., \( X \) is a set and \( \mathcal{O} \) is a family of subsets of \( X \) ("open sets") satisfying...
4. The usual axioms:

Then the \textbf{Borel} $\sigma$-algebra on $\mathbb{R}$, denoted $\mathcal{B}(\mathbb{R})$, is the smallest $\sigma$-algebra on $\mathbb{R}$ containing all open sets, i.e.,

$$\mathcal{B}(\mathbb{R}) = \mathcal{B}.$$

The elements in $\mathcal{B}(\mathbb{R})$ are called \textit{Borel sets} in $\mathbb{R}$.

So every open set is Borel; every closed set is countable union of Borel sets in $\mathbb{R}$.

Countable set-theoretic operations on Borel sets give Borel sets.

For ex. \( \emptyset \cup \text{set of cardinals} \in \text{Borel set in } \mathbb{R}. \) (Why?)

\textbf{Rem.:} Often it is desirable to generate a given $\sigma$-algebra by a suitable family of sets.

\textbf{Ex.} Let $\mathcal{R} = \{ \mathbb{R} \times \mathbb{R}^n : \mathbb{R} \text{ rectangle in } \mathbb{R} \}$, i.e.,

$$M = [a_1, b_1] \times \cdots \times [a_n, b_n] \in \mathcal{R}.$$ 

Then $\mathcal{B}(\mathbb{R}^n) = \mathcal{B}(\mathcal{R})$.

\textbf{Proof (outline)}:

\( R \in \mathcal{R} \) (why?), so $2(\mathcal{R}) \subseteq \mathcal{B}(\mathbb{R}^n)$. Every open set in $\mathbb{R}^n$ is a countable union of rectangles (exercise?). So, $\emptyset \subseteq 2(\mathcal{R})$, hence $2(\mathcal{R}) = 2(\emptyset) \subseteq 2(\mathbb{R})$. 
We conclude $\mu(\mathbb{R}) = \infty$.

**Def.** (Measures)

Let $X$ be a set equipped with a $\sigma$-algebra $\mathcal{A}$. A (positive) measure $\mu : (X, \mathcal{A}) \to [0, \infty]$ is a function $\mu : \mathcal{A} \to [0, \infty]$ such that:

1. $\mu(\emptyset) = 0$.
2. If $\{A_n : n \in \mathbb{N}\}$ is a countable disjoint collection in $\mathcal{A}$ (i.e., $A_n \cap A_m = \emptyset$ for $n \neq m$), then
   \[\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) \in [0, \infty].\]

(Countable additivity)

The pair $(X, \mathcal{A})$ is called a measurable space, and the triple $(X, \mathcal{A}, \mu)$ a measure space.

**Rem. 1.** Countable additivity implies finite additivity. If $\mathcal{A} = (X, \mathcal{A}, \mu)$ is a measure on $X$ and $\{A_1, \ldots, A_k\}$ is a finite disjoint collection in $\mathcal{A}$, then
\[
\mu\left(\bigcup_{k=1}^{k} A_k\right) = \sum_{k=1}^{k} \mu(A_k).
\]

Proof. Define $\emptyset = A_{k+1} = A_{k+2} = \cdots$ and use $\mu(\emptyset) = 0$ and count. additivity. QED.
6) Ex. Let $X = \mathbb{R}(X)$, $a \in X$

$$\mu = \delta_a$$

"Dirac measure at $a$".

$$\mathcal{S}_a(M) = \begin{cases} 0 & : \text{if } a \notin M \\ 1 & : \text{if } a \in M. \end{cases}$$

Ex. (Counting measure)

For all $\mathcal{U} \subseteq \mathbb{R}(X)$,

$$\mu(M) = \begin{cases} +\infty & : \text{if } M \text{ infinite} \\ \#M = \text{number of elements in } M & : \text{otherwise}. \end{cases}$$

Then $(X, \mathcal{U}, \mu)$ a measure space.

i) If $A, B \in \mathcal{U}$ and $A \subseteq B$, then

$$\mu(A) \leq \mu(B) \quad (\text{monotonicity})$$

ii) If $\{A_n\}_{n \in \mathbb{N}}$ then

$$\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) \quad (\text{subadditivity})$$

iii) If $A_n \subseteq X$ for we $n \in \mathbb{N}$ and $\bigcup A_n \uparrow$ (i.e., $A_n \subseteq A_{n+1}$ for $n \in \mathbb{N}$), then

$$\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \lim_{n \to \infty} \mu(A_n) \quad (\text{continuity from below})$$

iv) If $A_n \subseteq \mathcal{U}$ for we $n \in \mathbb{N}$ and $\bigcap A_n \uparrow$ (i.e., $A_n \supseteq A_{n+1}$ for $n \in \mathbb{N}$) and $\mu(A_n) < \infty$ for $n \in \mathbb{N}$, then

$$\mu\left(\bigcap_{n \in \mathbb{N}} A_n\right) = \lim_{n \to \infty} \mu(A_n) \quad (\text{continuity from above})$$

\[ \text{Q1.1.1)} \quad B = A \cup (B \setminus A) ; \quad A \cap (B \setminus A) = \emptyset \]

So,$$\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A).$$
11) \( B_n : = A_n \setminus \bigcup_{k=1}^{n-1} U A_k \) \( \subseteq A_n \).

Then \( U_{n \in N} B_n = U_{n \in N} A_n \) and \( \bigcap_{n \in N} B_n \). Pairwise disjoint.

So \( \mu \left( \bigcup_{n \in N} U A_n \right) = \mu \left( \bigcup_{n \in N} U B_n \right) \)

\[ = \sum_{n \in N} \mu (B_n) \leq \sum_{n \in N} \mu (A_n). \]

iii) If \( A_n \uparrow \), then \( B_n = A_n \setminus \bigcup_{k=1}^{n-1} U A_k = A_n \setminus A_{n-1} \), \( A_0 = \emptyset \).

So \( \mu \left( \bigcup_{n \in N} U A_n \right) = \mu \left( \bigcup_{n \in N} U B_n \right) \)

\[ = \lim_{N \to \infty} \sum_{n=1}^{N} \mu \left( A_n \setminus A_{n-1} \right) = \mu \left( \bigcup_{n \in N} A_n \right) \]

\[ = \lim_{N \to \infty} \mu (A_n). \]

iv) \( B_n : = A_1 \setminus A_n \to \mu (A_1) = \mu (A_1) + \mu (B_n) \)

\[ \bigcup_{n \in N} B_n = \bigcup_{n \in N} (A_1 \cap A_n^c) = A_1 \cap \bigcup_{n \in N} A_n^c \]

De Morgan \( A_1 \cap (\bigcap_{n \in N} A_n^c) = A_1 \setminus \bigcap_{n \in N} A_n \).

By iii)

\[ \mu (A_1) = \mu \left( \bigcap_{n \in N} A_n \right) = \mu \left( \bigcup_{n \in N} B_n \right) \]

\[ = \lim_{n \to \infty} \mu (B_n). \]
\[ \mathcal{G} = \ldots + \lim_{n \to \infty} \frac{1}{\mu(A_n)} \mu(A_n) \] so
\[ \mu(A) = \lim_{n \to \infty} \mu\left( \bigcap_{n=1}^{\infty} A_n \right) = \lim_{n \to \infty} \mu(A_n). \]

Since \( \mu(A) < \infty \) and \( \bigcap_{n=1}^{\infty} A_n = \lim_{n \to \infty} \mu(A_n). \]

\((X, \mathcal{A}, \mu)\) measure space

\(P(x)\) statement about point \(x \in X\)

(That may be true or false depending on \(x\)).

We say that \(P(x)\) is true for
\[ \mu \text{-a.e. } \] every \(x\) (for \(\mu \text{-a.e. } x\))
if there is a \( \delta \) s.t. \(E \in \mathcal{A}\) with
\[ \mu(E) = 0 \] (E is "an exceptional set")

\(P(x)\) is true for each \(x \in X \setminus E\).
(Statement is true for all points except for \(\mu\) pts. in a \(\delta\) s.t. \(\mu\) measure 0.

One would like to say:
\(P(x)\) holds for \(\mu\text{-a.e. } x\) if
\[ \mu\left( \{ x \in X : P(x) \text{ false} \} \right) = 0, \]
but \( \{ x \in X : P(x) \text{ false} \} \neq \emptyset \) if \(A \in \mathcal{A} \)

in general.

If \(P(x)\) true, then \(\mu\text{-a.e. } x\), then \(\exists x\).
\(E \in \mathcal{A}\) with \(\mu(E) = 0\). Assume \(F \in \mathcal{A}\).
A measure space \((X, \mathcal{A}, \mu)\) is called complete (otherwise, complete with respect to \(\mu\)) if subsets of \(\mu\)-null sets are \(\mu\)-null sets. I.e., if \(A, B \subseteq X\), and \(B \subseteq A\), then \(B \in \mathcal{A}\) (and so \(\mu(B) = \mu(A) = 0\), and \(\mu(B) = 0\)).

**Theorem:** \((X, \mathcal{A}, \mu)\) measure space, 
\[ \mathcal{N} = \{ N \in \mathcal{A} : \mu(N) = 0 \} \]
family of \(\mu\)-null sets, 
\[ \mathcal{U} = \{ A \cup B : A \in \mathcal{A}, \forall N (N \in \mathcal{N} s.t. B \subseteq N) \} \]
Then \(\mathcal{U}\) is a \(\sigma\)-algebra on \(X\) with \(\mathcal{A} \subseteq \mathcal{U}\), and over \(\mathcal{U}\) can uniquely be extended to a measure on \(\mathcal{U}\) s.t. \(\mathcal{N}\) is complete.

**Proof:** \(\mathcal{N}\) is closed under countable unions; so \(\mathcal{U}\) is closed under countable unions. Obviously, \(\mathcal{U} \supseteq \mathcal{A}\), and, 
\[ \phi, \overline{X} \in \mathcal{U}. \]
Let \(M \in \mathcal{U}\); then \(M = A \cup B\), where \(A \in \mathcal{A}\), i.e. \(N \in \mathcal{N} s.t. B \subseteq N\).
\[ A \cap N = \emptyset, \text{ otherwise, replace } A \text{ by } A \cap N \text{ and } B \text{ by } B \cup (A \cap N) \]

Then \[ M^c = (A \cup B)^c = A^c \cap B^c = (A \cap N^c) \cup (N \setminus B) \subseteq N, \quad \forall \in N \]

So, \( \mathcal{U} \) is \( \mathcal{F} \)-algeba.

Define \( \mathcal{F}(A \cup B) = \mathcal{F}(A) \)

\[ \text{if } A \cup B, B \in \mathcal{N}, N \subseteq N. \]

This is well-defined.

\[ A \cup B = A_1 \cup B_1, \quad A_1, A_2 \in \mathcal{N}, \quad B_1, B_2 \in \mathcal{N}_2, \]

where \( N, N_1, N_2 \subseteq N. \)

Then \( A_1 \subseteq A_2 \cup B_2 \subseteq A_2 \cup N \subseteq N_2 \}

\[ \mu(A_1) \leq \mu(A_2 \cup N \subseteq N_2) = \mu(A_2) + \mu(N \subseteq N_2) = \mu(A_2) + \mu(N) = \mu(A_2) \]

Similarly, \( \mu(A_2) \leq \mu(A_1) \) and \( \mu(A_1) = \mu(A_2) \).

It is easy to see \( \mathcal{N} \) is a \( \sigma \)-algebra, is complete, extends \( \mu \), and is uniquely determined with those properties.
Construction of non-trivial measures

premeasure $\rightarrow$ outer measure $\rightarrow$ measure.

**Def.** Let $\mathcal{X}$ be a set, $\mu^+: \mathcal{P}(\mathcal{X}) \rightarrow [0, \infty]$ is called an outer measure on $\mathcal{X}$ if

(i) $\mu^+(\emptyset) = 0$

(ii) $\mu^+(A) \leq \mu^+(B)$ whenever $A \subseteq B$ (monotonicity)

(iii) $\mu^+(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu^+(A_n)$, whenever $A_n \subseteq \mathcal{X}$, $n \in \mathbb{N}$. (Countable additivity).

Corollary: The (inner) version:
If $\mu^+$ is an outer measure on $\mathcal{X}$, then there exists a natural $\sigma$-algebra $\mathcal{A}$ (depending on $\mu^+$) s.t. $\mu = \mu^+|\mathcal{A}$ is a measure.

We define $\mathcal{A} \subseteq \mathcal{P}(\mathcal{X})$ as the set of all $A \subseteq \mathcal{X}$ s.t.

\[ \mu^+(T) = \mu^+(T \cap A) + \mu^+(T \cap A^c) \]

for all $T \subseteq \mathcal{X}$ (\textit{last sets}).

Note: The inequality is in $\mu$ is always true by (finite) subadditivity.

So $(\star)$ is equivalent to

\[ \mu^+(T) = \mu^+(T \cap A) + \mu^+(T \cap A^c) \]

for all $T \subseteq \mathcal{X}$. \( \blacksquare \)
12. Caratheodory's Theorem (precise version)

Let $\mu^*$ be a outer measure on $X$, and $\mathcal{U}$ be the family of all sets $A \subseteq X$ satisfying (*).

Then $\mathcal{U}$ is a $\sigma$-algebra on $X$ and $\mu = \mu^*|_\mathcal{U}$ is a complete measure.

For the proof, we need:

Def. (Algebra) A family $\mathcal{U}$ of subsets of $X$ is called an algebra on $X$ if

$\emptyset, X \in \mathcal{U}$, $A \in \mathcal{U}$ $\implies$ $A^c \in \mathcal{U}$ and $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{U}$.

Let $\mathcal{U}$ be an algebra on $X$, and suppose that $A \in \mathcal{U}$ implies $\bigcap_{n \in \mathbb{N}} A_n \in \mathcal{U}$. Then $\mathcal{U}$ is a $\sigma$-algebra.

Proof. Let $A, B \in \mathcal{U}$, $n \in \mathbb{N}$.

Define $B_n = A \cap B$, $n \in \mathbb{N}$.

Then $B_n \in \mathcal{U}$ and $U A_n = \bigcup_{n \in \mathbb{N}} B_n \in \mathcal{U}$ by hypotheses.

So $\mathcal{U}$ is closed under countable unions and $\sigma$-algebras.

Note: In the above, we can also require $A \in \mathcal{U}$, $n \in \mathbb{N}$, $A_n \in \mathcal{U}$, and $A_n \cap B \in \mathcal{U}$ for disjoint $A_n$ (i.e., $B \in \mathcal{U}$ for disjoint $A_n$).
Proof of Caratheodory's Theorem:

1. A consisting of all satisfying (**) or equilibria (***) is a \( \tau \)-algebra

We'll show that \( A \) is an algebra that is closed under monotonone limits.

(Levi \( \rightarrow U \) \( \tau \)-algebra)

(*) symmetric in \( A, A^c \), so

\[ A \cup A^c \]

Hence \( A \)

is closed under taking complements.

\( \phi \in \mathcal{X} \), because \( U \).

\[ \forall T \in \mathcal{X}, \mu^*(T \cap \phi) + \mu^*(T \cap \phi^c) = \mu^*(\phi) + \mu^*(T) = \mu^*(T). \]

\[ \mu^*(T \cap \phi^c) \]

\[ \subseteq \mu^*(T \cap (A \cup B)) + \mu^* \left( T \cap (A \cup B)^c \right) \]

\[ \subseteq \mu^*(T \cap (A \cap B)) + \mu^* \left( T \cap (A \cap B)^c \right) + \mu^* \left( T \cap (A^c \cup B) \right) \]

\[ \subseteq \mu^*(T \cap (A \cup B)) + \mu^* \left( T \cap (A \cup B)^c \right) \]

i.e.

\[ A \cup B \text{ solves } (**), \text{ and so } \mu^* \left( T \cap (A \cup B)^c \right). \]
15. Noted that on $\mathcal{A}$, $\mu^+$ is finitely additive,

if $A, B \in \mathcal{A}$, and $\emptyset \cap A \cap B = \emptyset$

$$\mu^+(A \cup B) = \mu^+(A \cup B \setminus (A \cup B^c)) = \mu^+(A) + \mu^+(B).$$

Let $A_n \in \mathcal{A}$ where $\mathcal{N}$, arbitrary and pairwise disjoint, $B_n = A_n \cup \ldots \cup A_n \in \mathcal{F}$

$B \in \mathcal{U}$.

WTS $B \in \mathcal{U}$, $\mu^+(\bigcup_{n=1}^{\infty} A_n) = \mu^+(B)$.

$$\mu^+(T \cap B_n) = \mu^+(T \cap B_n \cap A_n) = \mu^+(T \cap A_n) + \mu^+(T \cap B_n \setminus A_n)$$

So, inductively,

$$\mu^+(T \cap B_n) = \sum_{k=1}^{n} \mu^+(T \cap A_k)$$

Now, $B_n \in \mathcal{U}$

$$\mu^+(T) = \mu^+(T \cap B_n) + \mu^+(T \cap B_n^c)$$

$$\geq \sum_{k=1}^{\infty} \mu^+(T \cap A_k) + \mu^+(T \cap B_n^c)$$

Letting $n$ go to $\infty$, and using subadditivity,

$$\mu^+(T) = \geq \sum_{k=1}^{\infty} \mu^+(T \cap A_k) + \mu^+(T \cap B_n^c) = \mu^+(T \cap B) + \mu^+(T \cap B_n^c) \geq \mu^+(T\cap B)$$

$\Phi^3(\mu)$