Homework 9 (due: Fr, Dec. 1)

(1) We know that (standard) Brownian motion $\{B_t\}_{t\in[0,\infty)}$ is the scaling limit of simple random walk. More precisely, this means the following: suppose that X_i, $i \in \mathbb{N}$, are i.i.d. random variables such $\mathbb{P}(X_i = \pm 1) = 1/2$. Then for $t \in [0,\infty)$ we have the convergence in distribution

$$\frac{1}{\sqrt{n}}(X_1 + \cdots + X_{\lfloor nt \rfloor}) \Rightarrow B_t \quad \text{as } n \to \infty. \tag{1}$$

Write a computer program that gives two approximate plots of B_t for $t \in [0,T]$ based on the left hand side in formula (1) for the values $T = 5$, $n = 10$, and $T = 5$, $n = 200$. Attach a copy of your programming code and a plot of sample paths for these two pairs of (T, n).

Hint: Use your favorite programming language MATLAB, C++, Python, etc. I used Octave (freely available on the internet for Linux, Windows, Mac) to produce the sample paths that you can find on the course web-page.

To find the sample paths, fix T and n, and plot the points $(x_k, y_k) \in \mathbb{R}^2$, where $x_k = k/n$ and

$$y_k = \frac{1}{\sqrt{n}} \sum_{i=1}^{k} X_i$$

for $k = 0, \ldots, nT$, and "connect the dots" (often done automatically by plotting routines).

(2) The price S_t of a stock at time $t \geq 0$ is modeled by geometric Brownian motion given by

$$S_t = S_0 \exp(\sigma B_t - \tfrac{1}{2}\sigma^2 t), \tag{2}$$

where S_0 is the stock price at time $t = 0$, $\sigma \geq 0$ is a parameter measuring the volatility of the stock, and $\{B_t\}_{t\in[0,\infty)}$ is (standard) Brownian motion.

Based on Problem 1 write a program that simulates the process S_t in the interval $[0, 5]$, where we take $S_0 = 1$, $\sigma = 0.1$, and $n = 200$ for the parameter in (1). Attach a copy of your programming code and a plot of a sample path.

(3) The following table gives the prices for shares of Tesla Inc. (Ticker: TSLA) for the last few days at market close.

date	closing price
11/21	317.81
11/20	308.74
11/17	315.05
11/16	312.50
11/15	311.30
11/14	308.70
11/13	315.40
11/10	302.99
11/09	302.99
11/08	304.39
11/07	306.05
11/06	302.78

Use this data to compute the historical volatility σ_d of this stock per day for the period 11/06–11/21. Indicate the method how you obtained σ_d. What would be the corresponding volatility σ_a per year?

(4) We consider Twitter Inc. and the prices of the call options as in HW7, Problem (3). We denote by $S_0 = 19.90$ the price of the stock (on 11/3/17, the time under consideration) and by K the strike of the option. We assume that interest rates are negligible. According to the Black-Scholes model for option pricing the fair value c of a call option (on 11/3/17) is

$$c = S_0 N(x_+) - K N(x_-),$$

where N is the cumulative distribution function of the standard normal random variable and

$$x_\pm = \frac{\log(S_0/K) \pm \frac{1}{2}\alpha^2}{\alpha},$$

where α is a parameter independent of the option.

(a) Show that we have the approximation

$$N(x) \approx \tfrac{1}{2} + \frac{1}{\sqrt{2\pi}} x$$

for x close to 0.

(b) Based on the approximation in (a) use the price of the option with strike $K = 20$ at the money to find an approximate value for α that fits the data. Hint: Assume $S_0 = K$ for simplicity.

(c) Use Newton's method to find an improved value of α using $S_0 = 19.90$ and the price of the option with strike $K = 20$.

(d) Using the value for α found in (c) and the Black-Scholes formula, compile a table for the fair values of the options as in HW7, Problem (3). Compare with the real data!