**Homework 8** (due: We, Nov. 22)

(1) Suppose a hedge fund has developed a stock screening method that identifies start-up companies with exceptional growth prospects. Let us call these companies the (potential) "Big Winners" (think of Google, Amazon, or Apple). Let us make the following assumptions: Only 1% of the start-up companies are Big Winners. If a company is a Big Winner, then the screening method will identify it in 100% of the cases. If the company is not a Big Winner, then in 10% of the cases the method identifies it incorrectly as a Big Winner.

Suppose a start-up company has been identified as a Big Winner by the stock screening method. What is the probability (in absence of any other evidence) that the start-up company is really a Big Winner? Hint: Review conditional probability, in particular Bayes' formula.

(2) Let $(X, Y)$ be a pair of random variables with a joint Gaussian distribution. Suppose both random variables are centered and that $X$ has variance $s$ and $Y$ has variance $t$, where $0 < s \le t$.

(a) Show that $\mathbb{E}(XY) = s$ if and only $X$ and $Z = Y - X$ are independent. Hint: You have to show two implications: $\mathbb{E}(XY) = s$ implies that $X$ and $Z$ are independent and, conversely, the independence of $X$ and $Z$ implies that $\mathbb{E}(XY) = s$.

(b) Suppose that $\mathbb{E}(XY) = s$. What is the probability density function $p(x, y)$ of the joint distribution of $X$ and $Y$?

(3) If we model the price $S_t$ of a stock at time $t \ge 0$ by a geometric Brownian motion, then

$$S_t = S_0 \exp(\sigma B_t - \tfrac{1}{2}\sigma^2 t), \tag{1}$$

where $\sigma \ge 0$ is a parameter measuring the volatility of the stock and $\{B_t\}_{t\in[0,\infty)}$ is (standard) Brownian motion.

(a) We know that this model is compatible with risk-neutral valuation with interest rate $r = 0$. Explain why we then should have $\mathbb{E}(S_t) = S_0$.

(b) Verify $\mathbb{E}(S_t) = S_0$ by explicit computation based on formula (1) and the distribution of $B_t$.

(c) If the risk-free interest rate is the constant $r \ge 0$, then $S_0 = e^{-rt}\mathbb{E}(S_t)$ in risk-neutral valuation. Find a simple modification of the formula (1) that is compatible with this formula.

(4) We consider a Brownian motion $B_t$ for $t \geq 0$. Then the stochastic process defined as $X_0 = 0$ and $X_t = tB_{1/t}$ for $t > 0$ is a *version* of Brownian motion. This means that the distributions of the vectors $(X_{t_1}, \dots, X_{t_n})$ and $(B_{t_1}, \dots, B_{t_n})$ are the same whenever $n \in \mathbb{N}$ and $0 \leq t_1 < \cdots < t_n$. It also implies that the process has continuous sample paths $t \mapsto X_t$ almost surely (= with probability one).

(a) To make the statement about distributions plausible, determine the distribution of $X_t$ and determine $\mathrm{Cov}(X_t, X_s)$ for $s, t \geq 0$. What type of random variable is $X_t$ and what is its mean and variance?

(b) Use the process $X_t$ to show that

$$\lim_{t \to +\infty} \frac{B_t}{t} = 0$$

almost surely. In other words, the typical Brownian path grows at a sublinear rate.