Homework 7 (due: Mo, Nov. 13)

(1) A stock price is currentl \$40. It is know that at the end of the month it will either be \$42 or \$38. Suppose the risk-free rate is 8% per year with continuous compounding. What is the value of a (European) call option with strike price \$39?

(2) (a) We consider the price of a stock as a random variable S_T depending on time T. Let O be a call option with strike K and expiration T. We assume that interest rates are irrelevant. If we adopt risk-neutral valuation, what is the present fair value f of O in terms of S_T and K?

(b) A reasonable assumption for a small time interval T and a non-dividend paying stock with a sideways trend is that the total price change of the stock $S_T - S_0$ is a combination of many small additive changes that are i.i.d. and have mean 0. According to HW 3, Prob. 4, this suggests that $S_T - S_0$ should have the distribution of a normal random variable with mean 0. In other words, we may assume that

$$S_T = S_0 + \alpha X,$$

where $\alpha > 0$ is a parameter and X is a standard normal random variable.

Based on this model, show that the fair value f of the option is given by

$$f = \alpha g(c),$$

where $c = (K - S_0)/\alpha$ and g is the function in HW 3, Prob. 3(e).

(3) The last price for shares of Twitter Inc. (ticker symbol: TWTR) on Friday, 11/3/17, was \$19.90. We consider call options on TWTR with expiration 11/10/17. So these options are traded for only five more days. We want to see how the theoretical model in (2) fits with market data in this case. These options had the following last prices on 11/3/17 (shown is the average of bid and ask; spreads were typically between 0.02–0.03):

strike	price
19	1.03
19.5	0.66
20	0.39
20.5	0.25
21	0.15
21.5	0.11

(a) Which α in (2)(c) gives the price of the option with strike $K = 20$? Hint: Essentially, $c = 0$ here!

(b) Use this approximate value of α to compute c and $g(c)$ for the different strikes of the options. List your results in a table!

(c) According to 2(c) the price f of the option is $f = \alpha g(c)$. Use the values $g(c)$ computed in (b) and linear regression to find α that best fits the data.

(d) Use this new α to recompute the values $g(c)$ and list the theoretical values $\alpha g(c)$ in a table.

(e) Do the computed values agree with the market prices? Are there some noticeable discrepancies?

(4) We now analyze the option prices in (3) using binomial models. We assume for simplicity that interest rates are negligible (i.e. equal to 0) and that the "uptick" u and the "downtick" d of the stock price for each trading day have the form $u = 1 + \beta$ and $d = 1 - \beta$, where $\beta > 0$ is a small parameter.

(a) Show that the probabilities for an uptick or a downtick are both equal to $p = 1/2$ if we adopt risk-neutral valuation.

(b) The options in (3) have 5 trading days left before expiration. So there will be $n \in \{0, 1 \ldots, 5\}$ upticks of our stock price and $5 - n$ downticks. Express the probability that there are exactly n upticks and $5 - n$ downticks in terms of a binomial coefficient (and other factors).

(c) We may assume that there are at least 3 upticks for the price S_T of TWTR at expiration $T = 5$ days of the options to reach a value ≥ 20. Find an explicit formula for the expected value of the option with strike $K = 20$ at expiration in terms of a function of β. If we adopt risk-neutral valuation, this is also the present fair value of the option.

(d) Find the value of β that best matches the market price of the option with strike $K = 20$. Hint: Use Newton's method with the initial value $\beta = 0$.

(e) Use the value of β computed in (d) to find the fair values of the options with strikes $K = 20.5$ and $K = 21$.